GANDHI ACADEMY OF **TECHNOLOGY AND ENGINEERING**

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LECTURE NOTES ON ENGINEERING MATHEMATICS-II

CONTENTS

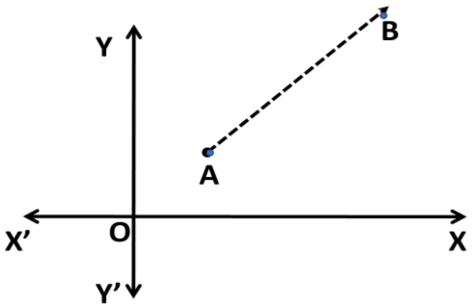
VECTOR

Introduction:
In physics or mechanics, while studying the motion you are using a variety of quantities such as distance, displacement, speed, velocity, acceleration, mass, force, momentum, work, power, energy etc. to describe the motion. These quantities can be classified into two categories 1) Scalar quantity 2) Vector quantity.

The quantities possessing a numerical value (or magnitude) are called as scalar quantities.
Examples: Distance, speed, mass, work etc.

The quantities possessing magnitude as well as direction are called as vector quantities.
Examples: displacement, velocity, acceleration, force, momentum etc.

Representation of vectors:



In a plane the vector directed from a point A to B is denoted as $\overrightarrow{AB}$.Here the point A is called as initial point and B is the terminal point or (Final point) .

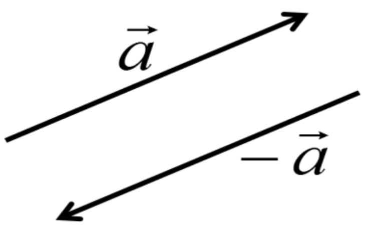
Notes: The length of a vector $\overrightarrow{AB}$ is the distance or magnitude of the vector $\overrightarrow{AB}$. It is denoted as $|\overrightarrow{AB}|$.
It is a scalar quantity , which is always positive.
Vectors are also denoted by using the lowercase bold alphabets **a, b, c** etc. or $\vec{a}, \vec{b}, \vec{c}$, etc.

Types of vectors:

Null vector:
Vector with magnitude zero (0) is called as a null vector or zero vector. It is written as $\vec{O}$.
Example: Every point is a null vector.

Unit vector: A vector with magnitude unity (1) is called as unit vector. If $\vec{a}$ be a vector, then the unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ and given by : $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Co-initial vectors: Vectors starting from a point are called as co-initial vectors.

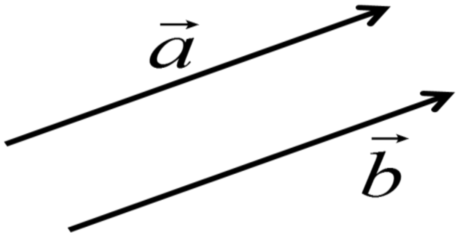


Negative of a vector: A vector having same magnitude but opposite direction is known as negative vector.

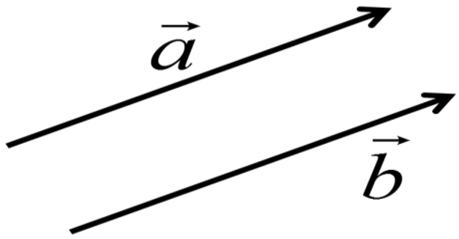
If $\vec{a}$ be a vector, then the negative of $\vec{a}$, written as -$\vec{a}$
Note: the magnitudes of $\vec{a}$ and -$\vec{a}$ are same. i.e. $|\vec{a}| = |-\vec{a}| = a$

Scalar multiplication of a vector:
Suppose k is a scalar quantity and $\vec{a}$ be a vector quantity, then the multiplication k$\vec{a}$ is a new vector called as scalar multiplication of $\vec{a}$. The vector k$\vec{a}$ is a vector whose length is $|k|$ times that of $\vec{a}$. The direction of k$\vec{a}$ will be same as of $\vec{a}$ if k is positive and opposite of $\vec{a}$ if k is negative.

Parallel vectors:
Two vectors are said to be parallel if both are of either same or opposite direction. In otherwords $\vec{a}$ and $\vec{b}$ will be parallel if $\vec{a}$ and $\vec{b}$ are scalar multiple of each other i.e. $\vec{a} = k\vec{b}$,where k is scalar.



Equal vectors:
Two vectors are said to be equal if they have same magnitude as well as same direction.



Like and Unlike vectors
Two vectors are said to be like if they are parallel but in same direction.
Two vectors are said to be unlike if they are parallel but in opposite direction.

Note: Like and Unlike vectors may be of same or different magnitudes.

Collinear of vectors:

Two vectors are said to be collinear if one is scalar multiple of other and they have a common point.
i.e $\vec{a} = k\vec{b}, where\ k\ is\ a\ scalar$

Collinear of three points:
Three points A,B and C are said to be collinear if they lie in same line and the condition is

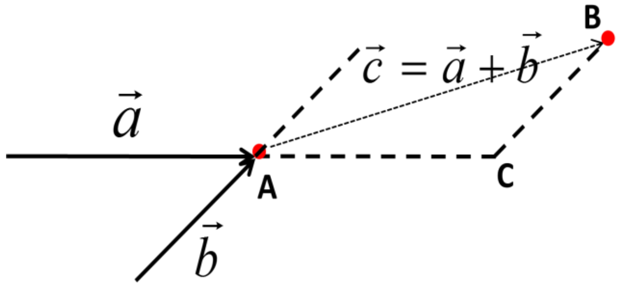
$$\overrightarrow{AB} = k\overrightarrow{AC}$$

Or

$$\overrightarrow{AB} = k\overrightarrow{BC}$$

Addition and Subtraction of vectors:

If $\overrightarrow{AC} = \vec{a}$ and $\overrightarrow{CB} = \vec{b}$ be any two vectors and join $\overrightarrow{AB}$.



Here the terminal point of $\overrightarrow{AC}$ is same as intial point of $\overrightarrow{CB}$. So $\overrightarrow{AB}$ is the sum of the vectors of $\overrightarrow{AC}$ and $\overrightarrow{CB}$.
i.e $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$=$\vec{a} + \vec{b}$, which is called triangle law of vector addition.

Subtraction of vectors:
The difference of two vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} - \vec{b}$ and which is defined as $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

Parallelogram law of vector addition:
Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ join $\overrightarrow{BC}$ and $\overrightarrow{AC}$ and make a parallelogram OACB. Join the diagonals $\overrightarrow{OC}$ and $\overrightarrow{BA}$.
Here $\overrightarrow{OA} = \vec{a} = \overrightarrow{BC}$ and $\overrightarrow{OB} = \vec{b} = \overrightarrow{AC}$
By using triangle law of vector addition we get $\overrightarrow{OC} = \vec{a} + \vec{b}$ and $\overrightarrow{BA} = \vec{a} - \vec{b}$

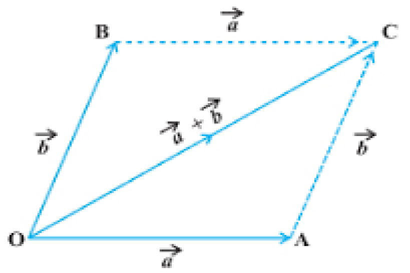


Fig. 5

Algebra of vectors:
1. Vector addition is commutative i.e. $\vec{a}$+$\vec{b}$= $\vec{b}$+$\vec{a}$
2. Vector addition is associative i.e. . $(\vec{a}+\vec{b}) + \vec{c}$= $\vec{a} + (\vec{b}+\vec{c})$
3. If m, n be any scalars, then $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$, $m(n\vec{a}) = n(m\vec{a}) = mn(\vec{a})$.
4. $1\vec{a} = \vec{a}$, $0\vec{a} = \vec{0}$

Position vector of a point in space:

If P be any point on the space and O be a fixed point (called origin) then the vector $\overrightarrow{OP} = \vec{r}$(say)
Is called the position vector of the point P with respect to origin O.
Which is written as P($\vec{r}$),(read as P be a point having position vector $\vec{r}$)

Note:
If A($\vec{a}$) and B($\vec{b}$) are two position vectors then $\overrightarrow{AB} = \vec{b} - \vec{a}$

Note:
If P(x,y) be any point on the plane and O be origin then

$$\overrightarrow{\boldsymbol{OP}} = \boldsymbol{x\hat{\imath} + y\hat{\jmath}}$$

Where $\hat{\imath}$ *and* $\hat{\jmath}$ are the unit vectors along the direction of X-axis and Y-axis.

Note:
If P(x,y,z) be any point on the space and O be origin then
$\overrightarrow{\boldsymbol{OP}} = \boldsymbol{x\hat{\imath} + y\hat{\jmath} + z\,\hat{k}}$

where, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unit vectors along X-axis, Y-axis and Z-axis respectively.

Note:
If $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ then $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$
And
$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

Vector joining two points:
The vector joining two points A(x_1, y_1, z_1) and B(x_2, y_2, z_3) can be obtained by using the triangle law of addition of vectors
i.e. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The coefficients x_2-x_1, y_2-y_1 and z_2-z_1 are called as the scalar components of the vector $\overrightarrow{AB}$ in the direction of X-axis, Y-axis and Z-axis respectively. And the vectors $(x_2 - x_1)\hat{\imath}, (y_2 - y_1)\hat{\jmath} and\ (z_2 - z_1)\hat{k}$ are called as vector components of the vector $\overrightarrow{AB}$ in the direction of X-axis, Y-axis and Z-axis respectively.

Problem: Find $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $5\vec{a}$ and $2\vec{a} - 7\vec{b}$, where $\vec{a} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$ and $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}$.

Example: Find the position vector of the point (2, 3, -5).
Ans: $2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$.

Example: Find the vector from the point (2, -3, 4) to (6, 4, 1) and hence find the modulus of this vector.

Example: Find the value of m if the modulus of the vector joining A(0, 1, -2) and B(-2, 3, m) is $\sqrt{8}$.

Example: Find the unit vector in the direction of the vector $2\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$.

Problem: Find the unit vector in the direction of the sum of the vectors $\vec{a}\ and\ \vec{b}$ where, $\vec{a} = -3\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$ and $\vec{b} = \hat{\imath} + 5\hat{\jmath} + 3\hat{k}$.
Example: Find the value of 'm' if the vectors $-3\hat{\imath} + m\hat{\jmath} - 8\hat{k}$ and $15\hat{\imath} + 2\hat{\jmath} + 40\hat{k}$ are parallel.
Example: Prove that the points (2, 1, -1), (3, -2, 1) and (8, -17, 11) are collinear.
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Questions carrying 2 marks

1. Find the position vector of the point A (2, -3). **Ans**: $2\hat{i} - 3\hat{j}$
2. Find the vector joining points A (1, -3) and B (-5, 4). **Ans**: $-6\hat{i} + 7\hat{j}$
3. Find the length of the vector joining P (2, -1) and Q (5,-4). **Ans** : $3\sqrt{2}$
4. Find the unit vector in the direction of the vector $\hat{i} + \hat{j} + \hat{k}$. **Ans** : $\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$
5. For what value of 'a' the vectors $2\hat{i} - 3\hat{j}$ and $a\hat{i} - 6\hat{j}$ are parallel. **Ans** : 4

6. If the vectors $\vec{a} = \alpha\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ are parallel, find the value $of\ \alpha$. **Ans**: α=-3

7. Find a unit vector parallel to the sum of vectors $3\hat{i} - 2\hat{j} + \hat{k}\ and\ -2\hat{i} + \hat{j} - 3\hat{k}$ **Ans**: $\frac{\hat{i}}{\sqrt{6}} - \frac{\hat{j}}{\sqrt{6}} - \frac{\hat{k}}{\sqrt{6}}$

<u>Question carrying 5 marks</u>

1. Prove by vector method that the points A(2,6,3), B(1,2,7) and C(3,10,-1) are collinear.
2. Prove that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \vec{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ form the sides of a right angled triangle.

Product of vectors:

The product of vectors can be defined using two special symbols i.e. (•) and (X).

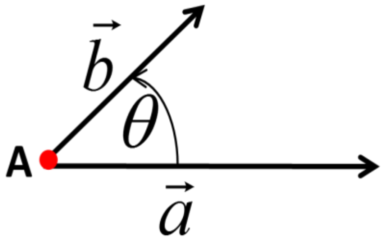
a. Scalar Product or Dot product

b. Vector Product or Cross Product

Scalar product (Dot product)

Definition:

Let $\vec{a}$ and $\vec{b}$ be any two coinitial vectors and θbe the angle between them measured from $\vec{a}$ to $\vec{b}$. Then the scalar product (or dot product) of $\vec{a}$ and $\vec{b}$ written as $\vec{a}\cdot\vec{b}$ and defined by



$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\theta = abcos\theta$$

Similarly

<u>Angle between two vectors</u>

Formula:

Angle between the vectors $\vec{a}$ and $\vec{b}$ is given by:

$$cos\theta = \pm\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \theta = cos^{-1}\left(\pm\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Condition-1:If two vectors are parallel then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| = ab$

Condition-2: If two vectors are perpendicular then $\vec{a}\cdot\vec{b} = 0 = \vec{b}.\vec{a}$

Properties of Dot product

a. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

b. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

c. If $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a}\cdot\vec{b} = 0$.

d. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive over addition)

Notes:

1. $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$ (Since $\hat{\imath},\hat{\jmath},\hat{k}$ are perpendicular)
2. $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$
3. If $\vec{a} = a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$, then $\vec{a}\cdot\vec{b} = a_1b_1+a_2b_2+a_3b_3$

Geometrical meaning of dot product:(Scalar and vector projection)

Scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Scalar projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

vector projection of $\vec{a}$ on $\vec{b} = \frac{(\vec{a}\cdot\vec{b})\vec{b}}{|\vec{b}|^2}$

vector projection of $\vec{b}$ on $\vec{a} = \frac{(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2}$

Work done:

$W = \vec{F} \cdot \vec{S}, where\ \vec{S} = \overrightarrow{AB}\ (\vec{F} = force)$

Example: Find the scalar product of the vectors $\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $-2\hat{\imath} + \hat{\jmath} - 3\hat{k}$
Example: Find the value of δ if the vectors $\hat{\imath} - \hat{\jmath} - 2\hat{k}$ and $\hat{\imath} + \delta\hat{\jmath} - 3\hat{k}$ are perpendicular to each other.
Example: Find the scalar projection and vector projection of $\vec{b}\ on\ \vec{a}$ where $\vec{a} = \hat{\imath} + 3\hat{\jmath} - 4\hat{k}, \vec{b} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$.
Example: Find the acute angle between the vectors $\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

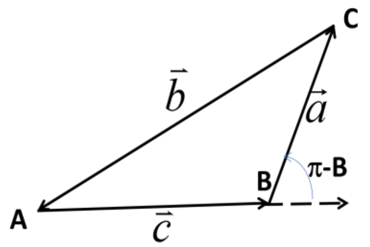
<u>Questions carrying 2 marks</u>

1. Find $\vec{a}.\vec{b}$ if (i) $\vec{a} = 2\hat{i} - 3\hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j}$,(ii) $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}\ and\ \vec{b} = \hat{i} - \hat{j} + \hat{k}$
2. Find the value of $'\lambda'$ if $\vec{a} = 6\hat{i} + \lambda\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$ are perpendicular to each other.
3. Show that the vectors $\vec{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - 3\hat{j} + \hat{k}$ are at right angles .

<u>Question carrying 5 marks</u>

1. Find the value of λ if $\vec{a} = (2, -2, 1)$ and $\vec{b} = (0, 2\lambda, 1)$ are perpendicular.
2. Find the angle between the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}\ and\ \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$
3. Find scalar and vector projections of $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ on $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$.
4. Find the work done by force $\vec{F} = 5\hat{i} - 2\hat{j} + 3\hat{k}$ which displaces a particle from A (1,-2,2) to B (3,1,5).
5. Prove by vector method that in any triangle ABC.

$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$



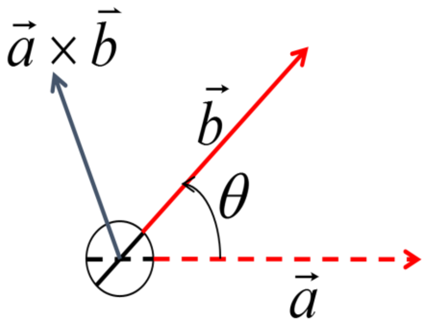
Proof: Assume the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to represent the sides of a triangle taken in order (Refer figure).

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$
$\Rightarrow \vec{a} = -(\vec{b} + \vec{c})$
$\Rightarrow \vec{a} \cdot \vec{a} = -(\vec{b} + \vec{c}) \cdot \vec{a} = (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c})$
$\Rightarrow \vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{b}$

6. Prove by vector method that an angle inscribed in a semicircle is right angle.
7. If the sum of two unit vectors is a unit vector, then prove that the magnitude of their difference will be $\sqrt{3}$.
8. In a triangle ABC prove by vector method that a=bcosC + c cosB

Vector Product (Cross Product)

Definition of vector product:



Let $\vec{a}$ and $\vec{b}$ be any two coinitial vectors and θbe the angle between them measured from $\vec{a}$ to $\vec{b}$. Then the vector product (or cross product) of $\vec{a}$ and $\vec{b}$ written as $\vec{a} \times \vec{b}$ and defined by

$\vec{a} \times \vec{b} = \left(|\vec{a}||\vec{b}|sin\theta\right)\hat{n} = (absin\theta)\hat{n}$

Here $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$. The direction of $\vec{a} \times \vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$. Hence $\hat{n}$ is in the direction of $\vec{a} \times \vec{b}$.

$|\vec{a} \times \vec{b}| = \left||\vec{a}||\vec{b}|sin\theta\hat{n}\right| = absin\theta\ |\hat{n}| = absin\theta$

Now $\vec{b} \times \vec{a} = \left(|\vec{a}||\vec{b}|\sin(-\theta)\right)\hat{n} = -absin\theta\ \hat{n}$

$|\vec{b} \times \vec{a}| = \left|-|\vec{b}||\vec{a}|sin\theta\hat{n}\right| = absin\theta\ |\hat{n}| = absin\theta$

So $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ but $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}|$

Note: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Sine of the angle between the vectors $\vec{a}$ and $\vec{b}$:

$sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

Condition-1: If two vectors are parallel then $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{a}| = 0\ or\ \vec{a} \times \vec{b} = \vec{0}$

Condition -2: If two vectors are perpendicular then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$=ab

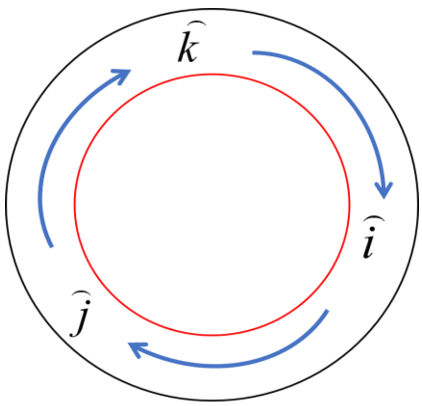
Properties of Cross product

a. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

b. $\vec{a} \times \vec{a} = \vec{0}$

c. $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Distributive over addition)

Notes:



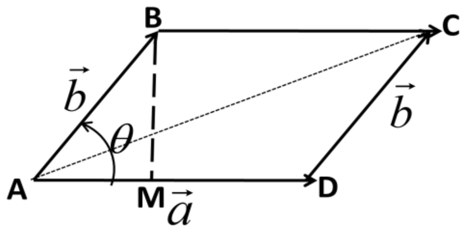
1. $\hat{\imath} \times \hat{\jmath} = \hat{k}$ (Since $\hat{\imath}, \hat{\jmath}, \hat{k}$ are perpendicular)
2. $\hat{\jmath} \times \hat{k} = \hat{\imath}$
3. $\hat{k} \times \hat{\imath} = \hat{\jmath}$
4. $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$
5. If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$,

then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= (a_2b_3 - b_2a_3)\hat{\imath} - (a_1b_3 - b_1a_3)\hat{\jmath} + (a_1b_2 - b_1a_2)\hat{k}$

Geometrical meaning of cross product:

Area of a parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$



If $\vec{a}$ and $\vec{b}$ represent the two sides of a parallelogram taken in order, then the area of the triangle will be $\frac{1}{2}|\vec{a}\times\vec{b}|$

Unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$:

The unit vector *perpendicular* to the vectors $\vec{a}$ *and* $\vec{b}$ will be $\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Note:The vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$ will be $\vec{a}\times\vec{b}$

Note:Area of a triangle with vertices A, B and C is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Example: Find the vector product of the vectors $\hat{\imath}-5\hat{\jmath}+2\hat{k}$ and $-\hat{\imath}+\hat{\jmath}-3\hat{k}$.

Ex: Find $\vec{a}\times\vec{b}$ where $\vec{a}=3\hat{\imath}+\hat{\jmath}+7\hat{k}$ and $\vec{b}=\hat{\imath}-3\hat{\jmath}+4\hat{k}$.

Example: Find the vector perpendicular to both $5\hat{\imath}-2\hat{\jmath}+3\hat{k}$ and $\hat{\imath}-3\hat{\jmath}+4\hat{k}$.

Example: Find the unit vector perpendicular to both $\hat{\imath}+2\hat{\jmath}$ and $\hat{\imath}+3\hat{\jmath}+\hat{k}$.

Example: Find the area of the parallelogram whose adjacent sides are $\hat{\imath}+2\hat{\jmath}-4\hat{k}$ and $\hat{\imath}-3\hat{\jmath}+\hat{k}$.

Example: Find the area of the triangle whose vertices are at A(1, -1, 3), B(2, 1, 0) and C(3, 1, -1)

Example: Find the sine of the angle between the vectors $3\hat{\imath}-4\hat{\jmath}$ and $6\hat{\imath}-2\hat{\jmath}+3\hat{k}$.

<u>Question carrying 2 marks</u>

1. Find the vector perpendicular to both of the vectors $\hat{i}+\hat{j}$ *and* $\hat{j}+\hat{k}$. **Ans**: $\hat{i}-\hat{j}+\hat{k}$
2. Find the unit vector perpendicular to both the vectors $\hat{i}+\hat{j}$ and $\hat{i}-\hat{k}$.

   Ans: $\frac{-1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$
3. Find the unit vector perpendicular to both the vectors $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+3\hat{k}$

   Ans $\frac{2}{\sqrt{110}}\hat{i}-\frac{9}{\sqrt{110}}\hat{j}-\frac{5}{\sqrt{110}}\hat{k}$
4. Find area of parallelogram whose adjacent sides are given by vectors

   $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$ **Ans**: $8\sqrt{3}$
5. Find area of the triangle whose sides are given by vectors $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

   Ans: $4\sqrt{3}$
6. Find area of the triangle whose vertices are A(1,-2,3), B(3,1,2), C(2,3,-1).

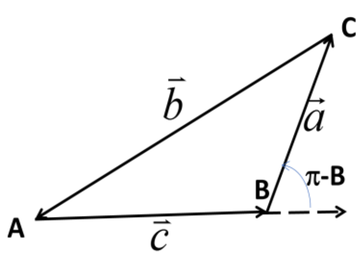
   Ans: $\frac{7\sqrt{3}}{2}$
7. Find sine angle between the vectors $\vec{a}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=\hat{i}+3\hat{j}+2\hat{k}$

   Ans: $\frac{\sqrt{171}}{14}$
8. Find the area of the parallelogram whose diagonals are $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=-3\hat{i}+4\hat{j}-\hat{k}$ **Ans**: $\frac{3\sqrt{30}}{2}$
9. Prove that $\left(\vec{a}\times\vec{b}\right)^2=a^2b^2-\left(\vec{a}.\vec{b}\right)^2$

10. Prove that by vector method in any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Hints: Assume the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to represent the sides of a triangle taken in order (Refer figure).



$\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} = -(\vec{b} + \vec{c})$ ------------ (a)

$\Rightarrow \vec{a} \times \vec{a} = -(\vec{b} + \vec{c}) \times \vec{a} = -(\vec{b} \times \vec{a}) - (\vec{c} \times \vec{a})$

$\Rightarrow \vec{0} = -(\vec{b} \times \vec{a}) - (\vec{c} \times \vec{a})$

$\Rightarrow (\vec{a} \times \vec{b}) = (\vec{c} \times \vec{a})$ ------------ (b)

Similarly, take the cross product with $\vec{b}$ on both sides of equation (a) and get

$\Rightarrow (\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c})$ ------------ (c)

Comparing Equation (b) and (c) in getting

$(\vec{a} \times \vec{b}) = (\vec{b} \times \vec{c}) = (\vec{c} \times \vec{a})$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

$\Rightarrow absin(\pi - C) = bcsin(\pi - A) = casin(\pi - B)$

$\Rightarrow absinC = bcsinA = casinB$

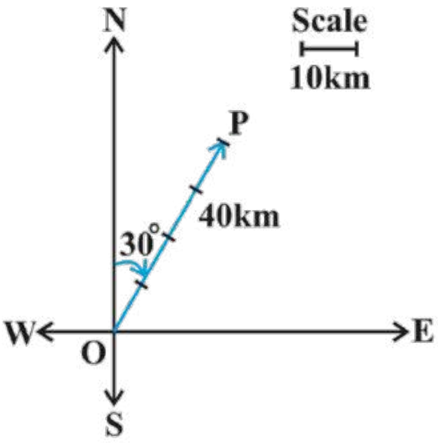
Now divide 'abc' on both sides to get the answer.

EXERCISE

Question 1:

Represent graphically a displacement of 40km, 30° east of north.

Solution:



$\overrightarrow{OP}$ represents the displacement of 40km, 30° north-east.

Question 2:

Classify the following measures as scalars and vectors.

(i) 10 kg (ii) 2 meters north-east (iii) 40°

(iv) 40 watt (v) 10^{-19} coulomb (vi) $20 m/s^2$

Solution:

(i) 10kg is a scalar.

(ii) 2 meters north-west is a vector.

(iii) 40° is a scalar.

(iv) 40 watts is a scalar.

(v) 10^{-19} Coulomb is a scalar.

(vi) $20 m/s^2$ is a vector

Question 3:

Classify the following as scalar and vector quantities.

(i) time period (ii) distance (iii) force

(iv) velocity (v) work done.

Solution:

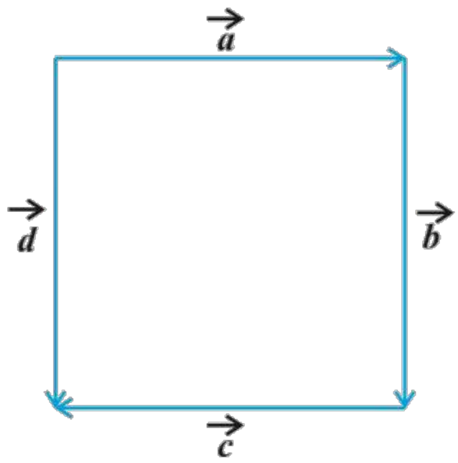
(i) Time period is a scalar.

(ii) Distance is a scalar.

(iii) Force is a vector.
(iv) Velocity is a vector.
(v) Work done is a scalar.

Question 4:

In figure, identify the following vectors.



(i) Coinitial (ii) Equal (iii) Collinear but not equal.

Solution:

(i) Vectors $\vec{a}$ and $\vec{d}$ are coinitial.
(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal.
(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal.

Question 5:

Answer the following as true or false.

(i) $\vec{a}$ and $-\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude.
(iii) Two vectors having same magnitude are collinear.
(iv) Two collinear vectors having the same magnitude are equal.

Solution:

(i) True.
(ii) False.
(iii) False.
(iv) False

EXERCISE

Question 1:

Compute the magnitude of the following vectors:

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$; $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$; $\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$

Solution:

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$|\vec{b}| = \sqrt{(2)^2 + (-7)^2 + (-3)^2} = \sqrt{4+49+9} = \sqrt{62}$$

$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

Question 2:

Write two different vectors having same magnitude.

Solution:

Let $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{b} = (2\hat{i} + \hat{j} - 3\hat{k})$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4+1+9} = \sqrt{14}$$

But $\vec{a} \neq \vec{b}$

Question 3:

Write two different vectors having same direction.

Solution:

Let $\vec{p} = (\vec{i} + \vec{j} + \vec{k})$ and $\vec{q} = (2\vec{i} + 2\vec{j} + 2\vec{k})$

The DCs of $\vec{p}$ are

$$l=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$
$$m=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$
$$n=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

The DCs of $\vec{q}$ are

$$l=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}$$
$$m=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}$$
$$n=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}$$

But $\vec{p}\neq\vec{q}$

Question 4:

Find the values of x and y so that the vectors $2\vec{i}+3\vec{j}$ and $x\vec{i}+y\vec{j}$ are equal.

Solution:

It is given that the vectors $2\vec{i}+3\vec{j}$ and $x\vec{i}+y\vec{j}$ are equal.

Therefore,

$$2\vec{i}+3\vec{j}=x\vec{i}+y\vec{j}$$

On comparing the components of both sides

$$\Rightarrow x=2$$
$$\Rightarrow y=3$$

Question 5:

Find the scalar and vector components of the vector with initial point $(2,1)$ and terminal point $(-5,7)$.

Solution:

Let the points be $P(2,1)$ and $Q(-5,7)$

$$\overrightarrow{PQ} = (-5-2)\hat{i} + (7-1)\hat{j}$$
$$= -7\hat{i} + 6\hat{j}$$

So, the scalar components are -7 and 6, and
the vector components are $-7\hat{i}$ and $6\hat{j}$.

Question 6:

Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} + 7\hat{k}$

Solution:

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} + 7\hat{k}$.

Therefore,

$$\vec{a} + \vec{b} + \vec{c} = (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k}$$
$$= 0\hat{i} - 4\hat{j} - \hat{k}$$
$$= -4\hat{j} - \hat{k}$$

Question 7:

Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Solution:

We have $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$

Hence,

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2}$$
$$= \sqrt{1+1+4}$$
$$= \sqrt{6}$$

Therefore,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}$$
$$= \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

Question 8:

Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points $(1,2,3)$ and $(4,5,6)$ respectively.

Solution:

We have the given points $P(1,2,3)$ and $Q(4,5,6)$

Hence,

$$\overrightarrow{PQ} = (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k}$$
$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$
$$\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + 3^2 + 3^2}$$
$$= \sqrt{9+9+9}$$
$$= \sqrt{27}$$
$$= 3\sqrt{3}$$

So, unit vector is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Question 9:

For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$.

Solution:

The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$

Therefore,

$$\vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k}$$
$$= 1\hat{i} + 0\hat{j} + 1\hat{k}$$
$$= \hat{i} + \hat{k}$$
$$\left|\vec{a} + \vec{b}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Thus, unit vector is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{\hat{i}+\hat{k}}{\sqrt{2}}$$
$$=\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$$

Question 10:

Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units.

Solution:

Let $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$

Hence,

$$|\vec{a}|=\sqrt{5^2+(-1)^2+(2)^2}$$
$$=\sqrt{25+1+4}$$
$$=\sqrt{30}$$

Therefore,

$$\hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}$$

Thus, a vector parallel to $5\hat{i}-\hat{j}+2\hat{k}$ with magnitude 8 units is

$$8\hat{a}=8\left(\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}\right)$$
$$=\frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}$$

Question 11:

Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

Solution:

We have $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}$

Now,

$$\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}$$
$$=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)$$
$$=-2\vec{a}$$

Since, $\vec{b} = \lambda \vec{a}$

Therefore, $\lambda = -2$

So, the vectors are collinear.

Question 12:

Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

Solution:

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

Therefore,

$$\begin{aligned} |\vec{a}| &= \sqrt{1^2 + 2^2 + 3^2} \\ &= \sqrt{1+4+9} \\ &= \sqrt{14} \end{aligned}$$

Thus, the DCs of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Question 13:

Find the direction of the cosines of the vectors joining the points $A(1,2,-3)$ and $B(-1,-2,1)$ directions from A to B.

Solution:

The given points are $A(1,2,-3)$ and $B(-1,-2,1)$.

Therefore,

$$\begin{aligned} \overrightarrow{AB} &= (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k} \\ &= -2\hat{i} - 4\hat{j} + 4\hat{k} \\ |\overrightarrow{AB}| &= \sqrt{(-2)^2 + (-4)^2 + 4^2} \\ &= \sqrt{4+16+16} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Thus, the DCs of $\overrightarrow{AB}$ are $\left(-\frac{2}{6},-\frac{4}{6},\frac{4}{6}\right)=\left(-\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$

Question 14:

Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axis OX, OY and OZ.

Solution:

Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$

Therefore,

$$|\vec{a}|=\sqrt{1^2+1^2+1^2}$$
$$=\sqrt{3}$$

Thus, the DCs of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Now, let α,β and γ be the angles formed by $\vec{a}$ with the positive directions of x,y and z axes respectively

Then,

$$\cos\alpha=\frac{1}{\sqrt{3}},\cos\beta=\frac{1}{\sqrt{3}},\cos\gamma=\frac{1}{\sqrt{3}}$$

Hence, the vector is equally inclined to OX, OY and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i}+2\hat{j}-\hat{k}$ and $-\hat{i}+\hat{j}+\hat{k}$ respectively, in the ratio $2:1$.

(i) Internally

(ii) Externally

Solution:

Position vectors of P and Q are given as:

$$\overrightarrow{OP}=\hat{i}+2\hat{j}-\hat{k} \text{ and } \overrightarrow{OQ}=-\hat{i}+\hat{j}+\hat{k}$$

(i) The position vector of R which divides the line joining two points P and Q internally in the ratio $2:1$ is

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1}$$
$$=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$
$$=\frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$$
$$=-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$$

(ii) The position vector of R which divides the line joining two points P and Q externally in the ratio $2:1$ is

$$\overrightarrow{OR}=\frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}$$
$$=\frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)}{1}$$
$$=-3\hat{i}+3\hat{k}$$

Question 16:

Find the position vector of the mid-point of the vector joining the points $P(2,3,4)$ and $Q(4,1,-2)$.

Solution:

The position vector of the mid-point R is

$$\overrightarrow{OR}=\frac{\left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\left(4\hat{i}+\hat{j}-2\hat{k}\right)}{2}$$
$$=\frac{(2+4)\hat{i}+(3+1)\hat{j}+(4-2)\hat{k}}{2}$$
$$=\frac{6\hat{i}+4\hat{j}+2\hat{k}}{2}$$
$$=3\hat{i}+2\hat{j}+\hat{k}$$

Question 17:

Show that the points A,B and C with position vectors, $\vec{a}=3\hat{i}-4\hat{j}-4\hat{k}, \vec{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-3\hat{j}-5\hat{k}$, respectively form the vertices of a right angled triangle.

Solution:

Position vectors of points A, B, and C are respectively given as:

$$\vec{a}=3\hat{i}-4\hat{j}-4\hat{k},\ \vec{b}=2\hat{i}-\hat{j}+\hat{k} \text{ and } \vec{c}=\hat{i}-3\hat{j}-5\hat{k}$$

Therefore,

$$\begin{aligned}\overrightarrow{AB}&=\vec{b}-\vec{a}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}\\&=-\hat{i}+3\hat{j}+5\hat{k}\\\overrightarrow{BC}&=\vec{c}-\vec{b}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}\\&=-\hat{i}+2\hat{j}-6\hat{k}\\\overrightarrow{CA}&=\vec{a}-\vec{c}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}\\&=2\hat{i}-\hat{j}+\hat{k}\end{aligned}$$

Now,

$$\begin{aligned}|\overrightarrow{AB}|^2&=(-1)^2+3^2+5^2=1+9+25=35\\|\overrightarrow{BC}|^2&=(-1)^2+(-2)^2+(-6)^2=1+4+36=41\\|\overrightarrow{CA}|^2&=2^2+(-1)^2+1^2=4+1+1=6\end{aligned}$$

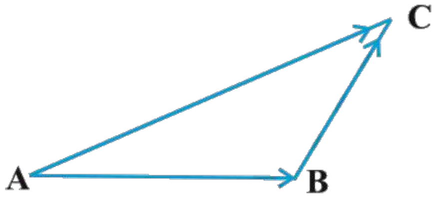
Also,

$$\begin{aligned}|\overrightarrow{AB}|^2+|\overrightarrow{CA}|^2&=35+6\\&=41\\&=|\overrightarrow{BC}|^2\end{aligned}$$

Thus, ABC is a right-angled triangle.

Question 18:

In triangle ABC which of the following is not true.



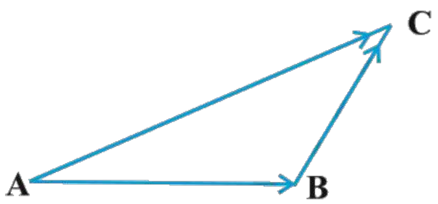
(A) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$

(B) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

(C) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$

(D) $\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

Solution:



On applying the triangle law of addition in the given triangle, we have:

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad \ldots(1)$$
$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$$
$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0 \qquad \ldots(2)$$

Hence, the equation given in option A is true.

Now, from equation (2)

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$
$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = 0$$

Hence, the equation given in option B is true.

Also,

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$
$$\Rightarrow \overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = 0$$

Hence, the equation given in option D is true

Now, consider the equation given in option C,

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = 0$$
$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA} \qquad \ldots(3)$$

From equations (1) and (2)

$$\Rightarrow \overrightarrow{AC} = \overrightarrow{CA}$$
$$\Rightarrow \overrightarrow{AC} = -\overrightarrow{AC}$$
$$\Rightarrow \overrightarrow{AC} + \overrightarrow{AC} = 0$$
$$\Rightarrow 2\overrightarrow{AC} = 0$$
$$\Rightarrow \overrightarrow{AC} = 0$$

Which is not true. So, the equation given in option C is incorrect.

Thus, the correct option is C.

Question 19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect?

(A) $\vec{b} = \lambda \vec{a}$, for some scalar λ

(B) $\vec{a} = \pm \vec{b}$

(C) the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

(D) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

If $\vec{a}$ and $\vec{b}$ are collinear vectors, they are parallel.

Therefore, for some scalar λ

$$\vec{b} = \lambda \vec{a}$$

If $\lambda = \pm 1$, then $\vec{a} = \pm \vec{b}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then,

$$\Rightarrow \vec{b} = \lambda \vec{a}$$
$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$
$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

Comparing the components of both the sides

$$\Rightarrow \vec{b_1} = \lambda a_1$$
$$\Rightarrow \vec{b_2} = \lambda a_2$$
$$\Rightarrow \vec{b_3} = \lambda a_3$$

Therefore,

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

However, $\vec{a}$ and $\vec{b}$ may have different directions.

Hence, that statement given in D is incorrect.

Thus, the correct option is D.

EXERCISE

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitude $\sqrt{3}$ and 2, respectively have $\vec{a}.\vec{b}=\sqrt{6}$

Solution:

It is given that

$$\left|\vec{a}\right|=\sqrt{3}$$
$$\left|\vec{b}\right|=2$$
$$\vec{a}.\vec{b}=\sqrt{6}$$

Therefore,

$$\Rightarrow \sqrt{6}=\sqrt{3}\times 2\cos\theta$$
$$\Rightarrow \cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$
$$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$$
$$\Rightarrow \theta=\frac{\pi}{4}$$

Question 2:

Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

Solution:

Let $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$.

Hence,

$$\left|\vec{a}\right|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$
$$\left|\vec{b}\right|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$
$$\vec{a}.\vec{b}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}-2\hat{j}+\hat{k}\right)$$
$$=1\times 3+(-2)(-2)+3\times 1$$
$$=3+4+3$$
$$=10$$

Therefore,

$$\Rightarrow 10 = \sqrt{14}\sqrt{14}\cos\theta$$
$$\Rightarrow \cos\theta = \frac{10}{14}$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$

Solution:

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{1}{\sqrt{1+1}}\{(1)(1)+(-1)1\}$$
$$= \frac{1}{\sqrt{2}}(1-1)$$
$$= 0$$

Question 4:

Find the projection of vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$

Solution:

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{(1)(7)+3(-1)+7(8)\}$$
$$= \frac{1}{\sqrt{49+1+64}}(7-3+56)$$
$$= \frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector which are manually perpendicular to each other.

$\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right), \frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right), \frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$

Solution:
Let

$$\vec{a}=\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right)=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$
$$\vec{b}=\frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right)=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}$$
$$\vec{c}=\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}$$

Now,

$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$
$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$
$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

So, each of the vector is a unit vector.

Hence,

$$\vec{a}.\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(-\frac{6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$
$$\vec{b}.\vec{c}=\frac{3}{7}\times\frac{6}{7}+\frac{2}{7}\times\left(-\frac{6}{7}\right)+\left(-\frac{3}{7}\right)\times\frac{2}{7}=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$
$$\vec{c}.\vec{a}=\frac{2}{7}\times\frac{6}{7}+\frac{3}{7}\times\frac{2}{7}+\frac{6}{7}\times\left(-\frac{3}{7}\right)=\frac{12}{49}+\frac{6}{49}+\frac{18}{49}=0$$

So, the vectors are mutually perpendicular to each other.

Question 6:
Find $|\vec{a}|$ and $|\vec{b}|$, if $\left(\vec{a}+\vec{b}\right)\left(\vec{a}-\vec{b}\right)=8$ and $|\vec{a}|=8|\vec{b}|$

Solution:

It is given that $\left(\vec{a}+\vec{b}\right)\left(\vec{a}-\vec{b}\right)=8$ and $|\vec{a}|=8|\vec{b}|$
Therefore,

$$\Rightarrow \left(\vec{a}+\vec{b}\right)\left(\vec{a}-\vec{b}\right)=8$$

$$\Rightarrow \vec{a}.\vec{a}-\vec{a}\vec{b}+\vec{b}\vec{a}-\vec{b}\vec{b}=8$$

$$\Rightarrow \left|\vec{a}\right|^2-\left|\vec{b}\right|^2=8$$

$$\Rightarrow \left(8\left|\vec{b}\right|\right)^2-\left|\vec{b}\right|^2=8$$

$$\Rightarrow 64\left|\vec{b}\right|^2-\left|\vec{b}\right|^2=8$$

$$\Rightarrow 63\left|\vec{b}\right|^2=9$$

$$\Rightarrow \left|\vec{b}\right|^2=\frac{8}{63}$$

$$\Rightarrow \left|\vec{b}\right|=\sqrt{\frac{8}{63}}$$

$$\Rightarrow \left|\vec{b}\right|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

Now,

$$\left|\vec{a}\right|=8\left|\vec{b}\right|$$

$$=\frac{8\times 2\sqrt{2}}{3\sqrt{7}}$$

$$=\frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $\left(3\vec{a}-5\vec{b}\right)\left(2\vec{a}+7\vec{b}\right)$

Solution:

$$\left(3\vec{a}-5\vec{b}\right)\left(2\vec{a}+7\vec{b}\right)=3\vec{a}.2\vec{a}+3\vec{a}.7\vec{b}-5\vec{b}.2\vec{a}-5\vec{b}.7\vec{b}$$

$$=6\vec{a}\vec{a}+21\vec{a}\vec{b}-10\vec{a}\vec{b}-35\vec{b}\vec{b}$$

$$=6\left|\vec{a}\right|^2+11\vec{a}\vec{b}-35\left|\vec{b}\right|^2$$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that angle between them is $60°$ and their scalar product is $\frac{1}{2}$

Solution:

Let θ be the angle between $\vec{a}$ and $\vec{b}$

It is given that $|\vec{a}| = |\vec{b}|$, $\vec{a}.\vec{b} = \frac{1}{2}$ and $\theta = 60°$

Therefore,

$$\Rightarrow \frac{1}{2} = |\vec{a}||\vec{b}|\cos 60°$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12$

Solution:

$$\Rightarrow (\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12$$

$$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}\vec{x} - \vec{a}\vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\Rightarrow |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Solution:

We have $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$

Then,

$$\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$$

$$= (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

Now,

$$\Rightarrow (\vec{a}+\lambda\vec{b}).\vec{c}=0$$
$$\Rightarrow \left[(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right].(3\hat{i}+\hat{j})=0$$
$$\Rightarrow 3(2-\lambda)+(2+2\lambda)+0(3+\lambda)=0$$
$$\Rightarrow 6-3\lambda+2+2\lambda=0$$
$$\Rightarrow -\lambda+8=0$$
$$\Rightarrow \lambda=8$$

Question 11:

Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any non-zero vectors $\vec{a}$ and $\vec{b}$.

Solution:

$$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)=|\vec{a}|^2\,\vec{b}.\vec{b}-|\vec{a}||\vec{b}|\vec{b}.\vec{a}+|\vec{b}||\vec{a}|\vec{a}.\vec{b}-|\vec{b}|^2\,\vec{a}.\vec{a}$$
$$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$$
$$=0$$

Question 12:

If $\vec{a}\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded above the vector $\vec{b}$?

Solution:

We have $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

Hence,

$$\Rightarrow |\vec{a}|^2=0$$
$$\Rightarrow |\vec{a}|=0$$

Therefore, $\vec{a}$ is the zero vector

Thus, any vector $\vec{b}$ can satisfy $\vec{a}.\vec{b}=0$

Question 13:

If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

Solution:

We have $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$

Therefore,

$$\left|\vec{a}+\vec{b}+\vec{c}\right|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)$$
$$0=\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+\left|\vec{c}\right|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$
$$0=1+1+1+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$$
$$\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\frac{-3}{2}$$

Question 14:

If either vector $\vec{a}=0$ or $\vec{b}=0$, then $\vec{a}.\vec{b}=0$. But the converse need not be true. Justify the answer with an example.

Solution:

Let $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$

Therefore,

$$\vec{a}.\vec{b}=2(3)+4(3)+3(-6)$$
$$=6+12-18$$
$$=0$$

Now,

$$\left|\vec{a}\right|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$$
$$\Rightarrow \vec{a}\neq 0$$
$$\left|\vec{b}\right|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$$
$$\Rightarrow \vec{b}\neq 0$$

So, the converse of the statement need not to be true.

Question 15:

If the vertices A, B, C of a triangle ABC are $(1,2,3),(-1,0,0),(0,1,2)$ respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Solution:

Vertices of the triangle are $A(1,2,3)$, $B(-1,0,0)$ and $C(0,1,2)$.

Hence,

$$\overrightarrow{BA} = \{1-(1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k}$$
$$= 2\hat{i} + 2\hat{j} + 3\hat{k}$$
$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k}$$
$$= \hat{i} + \hat{j} + 2\hat{k}$$
$$\overrightarrow{BA}.\overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right)\left(\hat{i} + \hat{j} + 2\hat{k}\right)$$
$$= 2\times1 + 2\times1 + 3\times2$$
$$= 2+2+6$$
$$= 10$$
$$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2}$$
$$= \sqrt{4+4+9}$$
$$= \sqrt{17}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2}$$
$$= \sqrt{6}$$
$$\overrightarrow{BA}.\overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\angle ABC)$$

Therefore,

$$\Rightarrow 10 = \sqrt{17}\times\sqrt{6}(\cos\angle ABC)$$
$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$
$$\Rightarrow (\angle ABC) = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Question 16:

Show that the points $A(1,2,7)$, $B(2,6,3)$ and $C(3,10-1)$ are collinear.

Solution:

The given points are $A(1,2,7)$, $B(2,6,3)$ and $C(3,10-1)$.

Hence,

$$\overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$
$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$
$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = 2\sqrt{33}$$

Therefore,

$$\begin{aligned} \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right| &= \sqrt{33} + \sqrt{33} \\ &= 2\sqrt{33} \\ &= \left|\overrightarrow{AC}\right| \end{aligned}$$

Hence, the points are collinear.

Question 17:

Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

Solution:

Let $\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overrightarrow{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\overrightarrow{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Hence,

$$\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$
$$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$
$$\overrightarrow{AC} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$
$$\left|\overrightarrow{AB}\right| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$
$$\left|\overrightarrow{BC}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$$

Therefore,

$$\begin{aligned} \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{AC}\right|^2 &= 6 + 35 \\ &= 41 \\ &= \left|\overrightarrow{AB}\right|^2 \end{aligned}$$

Thus, ΔABC is a right-angled triangle.

Question 18:

If $\vec{a}$ is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is a unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Solution:

$\Rightarrow |\lambda\vec{a}| = 1$

$\Rightarrow |\lambda||\vec{a}| = 1$

$\Rightarrow |\vec{a}| = \frac{1}{|\lambda|}$

$\Rightarrow a = \frac{1}{|\lambda|}$

Hence the correct option is D.

EXERCISE

Question 1:

Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

We have, $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Hence,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)$$

$$=19\hat{j}+19\hat{k}$$

Therefore,

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{(19)^2+(19)^2}$$

$$=\sqrt{2\times(19)^2}$$

$$=19\sqrt{2}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Solution:

We have $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Hence,

$$\vec{a}+\vec{b}=4\hat{i}+4\hat{j}$$

$$\vec{a}-\vec{b}=2\hat{i}+4\hat{k}$$

Therefore,

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}$$

$$=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)$$

$$=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{16^2+(-16)^2+(-8)^2}=\sqrt{2^2\times8^2+2^2\times8^2+8^2}$$

$$=8\sqrt{2^2+2^2+1}=8\sqrt{9}$$

$$=8\times3=24$$

So, the unit vector is

$$\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$

$$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}$$

$$=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

Question 3:

If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$.

Solution:

Let the unit vector $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Then, $\left|\vec{a}\right|=1$

Now,

$$\cos\frac{\pi}{3}=\frac{a_1}{|\vec{a}|}\Rightarrow a_1=\frac{1}{2}$$

$$\cos\frac{\pi}{4}=\frac{a_2}{|\vec{a}|}\Rightarrow a_2=\frac{1}{\sqrt{2}}$$

$$\cos\theta=\frac{a_3}{|\vec{a}|}\Rightarrow a_3=\cos\theta$$

Therefore,

$$\Rightarrow \sqrt{a_1^2+a_2^2+a_3^2}=1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta=1$$

$$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\theta=1$$

$$\Rightarrow \frac{3}{4}+\cos^2\theta=1$$

$$\Rightarrow \cos^2\theta=1-\frac{3}{4}=\frac{1}{4}$$

$$\Rightarrow \cos\theta=\frac{1}{2}$$

$$\Rightarrow \theta=\frac{\pi}{3}$$

Hence,

$$a_3=\cos\frac{\pi}{3}=\frac{1}{2}$$

So, $\theta=\frac{\pi}{3}$ and components of $\vec{a}$ are $\left(\frac{1}{2},\frac{1}{\sqrt{2}},\frac{1}{2}\right)$

Question 4:

Show that $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

Solution:

$$LHS = (\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$$
$$= (\vec{a}-\vec{b})\times\vec{a}+(\vec{a}-\vec{b})\times\vec{b}$$
$$= \vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b}$$
$$= 0+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-0$$
$$= 2\vec{a}\times\vec{b}$$
$$= RHS$$

Question 5:

Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=0$

Solution:

We have $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=0$

Therefore,

$$\Rightarrow (2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=0$$
$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$
$$\Rightarrow \hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$$6\mu-27\lambda=0$$
$$2\mu-27=0$$
$$2\lambda-6=0$$

Now,

$$2\lambda-6=0\Rightarrow\lambda=3$$
$$2\mu-27=0\Rightarrow\mu=\frac{27}{2}$$

Question 6:

Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=0$. What can you conclude about $\vec{a}$ and $\vec{b}$?

Solution:

When $\vec{a}.\vec{b}=0$

Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

Or $\vec{a} \perp \vec{b}$ (if $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$)

When $\vec{a} \times \vec{b} = 0$

Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

Or $\vec{a} \parallel \vec{b}$ (if $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$)

Since, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

So, $\vec{a} = 0$ or $\vec{b} = 0$.

Question 7:

Let the vectors $\vec{a}, \vec{b}, \vec{c}$ given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Solution:

We have

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Then,

$$\left(\vec{b} + \vec{c}\right) = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

Now,

$$\vec{a} \times \left(\vec{b} + \vec{c}\right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} = \begin{pmatrix} \hat{i}\left[a_2(b_3 + c_3) - a_3(b_2 + c_2)\right] - \hat{j}\left[a_1(b_3 + c_3) - a_3(b_1 + c_1)\right] \\ + \hat{k}\left[a_1(b_2 + c_2) - a_2(b_1 + c_1)\right] \end{pmatrix}$$

$$= \begin{pmatrix} \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] + \hat{j}\left[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1\right] \\ + \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right] \end{pmatrix}$$

Also,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$
$$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[a_3b_1-a_1b_3\right]+\hat{k}\left[a_2b_2-a_2b_1\right]$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$
$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_2c_2-a_2c_1\right]$$

Therefore,

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\begin{pmatrix}\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[a_3b_1-a_1b_3\right]+\hat{k}\left[a_2b_2-a_2b_1\right]\\ +\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_2c_2-a_2c_1\right]\end{pmatrix}$$
$$=\begin{pmatrix}\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]\\ +\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]\end{pmatrix}$$

Thus,

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence proved.

Question 8:

If either $\vec{a}=0$ or $\vec{b}=0$, then $\vec{a}\times\vec{b}=0$. Is the converse true? Justify your answer with an example.

Solution:

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$

Therefore,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}$$
$$=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)$$
$$=0$$

Now,

$$\left|\vec{a}\right|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

Thus,

$$\vec{a} \neq 0$$

Also,

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

Thus,

$$\vec{b} \neq 0$$

Hence, converse of the statement need not to be true.

Question 9:

Find the area of triangle with vertices $A(1,1,2)$ $B(2,3,5)$ and $C(1,5,5)$.

Solution:

Vertices of the triangle are $A(1,1,2)$ $B(2,3,5)$ and $C(1,5,5)$

Hence,

$$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k}$$
$$= \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k}$$
$$= -\hat{i} + 2\hat{j}$$

Therefore,

Area of the triangle $ar(\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

Now,

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix}$$
$$= \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2)$$
$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$
$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2}$$
$$= \sqrt{36 + 9 + 16}$$
$$= \sqrt{61}$$

Therefore,

$$ar(\Delta ABC)=\frac{1}{2}\sqrt{61}$$
$$=\frac{\sqrt{61}}{2}$$

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Solution:

We have $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

Hence,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}$$
$$=\hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2)$$
$$=20\hat{i}+5\hat{j}-5\hat{k}$$
$$\left|\vec{a}\times\vec{b}\right|=\sqrt{20^2+5^2+5^2}$$
$$=\sqrt{400+25+25}$$
$$=15\sqrt{2}$$

Thus, the area of parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $\left|\vec{a}\right|=3$ and $\left|\vec{b}\right|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

We have $\left|\vec{a}\right|=3$, $\left|\vec{b}\right|=\frac{\sqrt{2}}{3}$ and $\left|\vec{a}\times\vec{b}\right|=1$

Therefore,

$$\Rightarrow \left| |\vec{a}||\vec{b}|\sin\theta \right| = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Question 12:

Area of the rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution:

We have vertices $A\left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$, $B\left(\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$, $C\left(\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$ and $D\left(-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}\right)$.

Therefore,

$$\overrightarrow{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overrightarrow{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

Now,

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \hat{k}(-2)$$

$$= -2\hat{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \sqrt{(-2)^2}$$

$$= 2$$

So, area of the rectangle is 2 square units.

.LIMIT OF A FUNCTION

Lets discuss what a function is

A function is basically a rule which associates an element with another element.

There are different rules that govern different phenomena or happenings in our day to day life.

For example,

i. Water flows from a higher altitude to a lower altitude
ii. Heat flows from higher temperature to a lower temperature.
iii. External force results in change state of a body(Newton's 1st Rule of motion) etc.

All these rules associates an event or element to another event or element, say , x with y.

Mathematically we write,

y = f(x)

i.e. given the value of x we can determine the value of y by applying the rule 'f'

for example,

$$y = x + 1$$

i.e we calculate the value of y by adding 1 to value of x. This is the rule or function we are discussing.

Since we say a function associates two elements, x and y we can think of two sets A and B such that x is taken from set A and y is taken from set B. Symbolically we write

x ε A (x belongs to A)

y ε B (x belongs to B)

y = f(x) can also be written as

(x,y) ε f

Since (x,y) represents a pair of elements we can think of these in relations

f $\subseteq$ A X B or

f can thought of as a sub set of the product of sets A and B we have earlier referred to.

And, therefore, the elements of f are pair of elements like (x,y).

In the discussion of a function we must consider all the elements of set A and see that no x is associated with two different values of y in the set B

What is domain of function

Since function associates elements x of A to elements y of B and function must take care of all the elements of set A we call the set A as domain of the function. We must take note of the fact that if the function can not be defined for some elements of set A , the domain of the function will be a subset of A.

Example 1

Let A = $\{1,2,3,4,-1,0,-4\}$

B = $\{0,1,2,3,4,-1,-2,-3\}$

The function is given by

y = f(x) = x + 1

for x=1, y= 2

x=2, y=3

x=3,y=4

x=4,y=5

x=-1,y=0

x=0,y=1

x=-4,y=-3

clearly y=5 and y= -3 do not belong to set B. therefore we say the domain of this function is

the set $\{0,1,2,3,-1,\}$ which is a sub set of set A.

What is range of a function

Range of the function is the set of all y's whose values are calculated by taking all the values of x in the domain of the function. Since the domain of the function is either is equal to A or sub set of set A, range of the function is either equal to set b or sub set of set B.

In the earlier example,

Range of function is the set $\{1,2,3,4,0\}$ which is a sub set of set B

SOME FUNDAMENTAL FUNCTIONS

Constant Function

Y = f(x)=K, for all x

The rule here is: the value of y is always k, irrespective of the value of x

This is a very simple rule in the sense that evaluation of the value of y is not required as it is already given as k

Domain of 'f' is set of all real numbers

Range of 'f' is the singleton set containing 'k' alone.

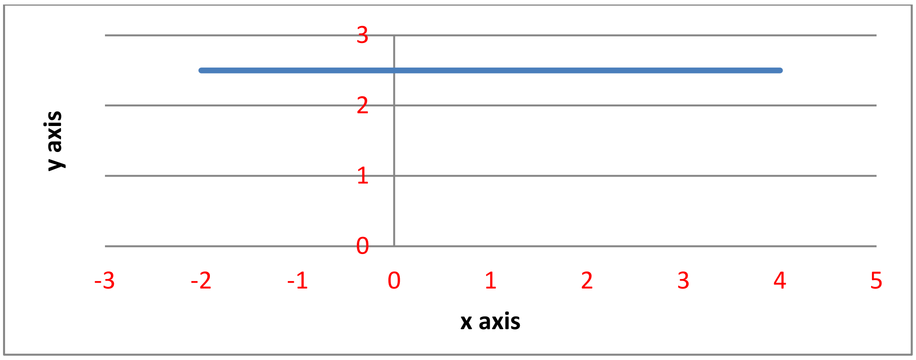
Or

Dom= R, set of all real numbers

Range= {k}

Graph of Constant Function

Let y = f(x) = k =2.5



The graph is a line parallel to axis of x

Identity Function

Y = f(x)=x, for all x

The rule here is: the value of y is always equals to x

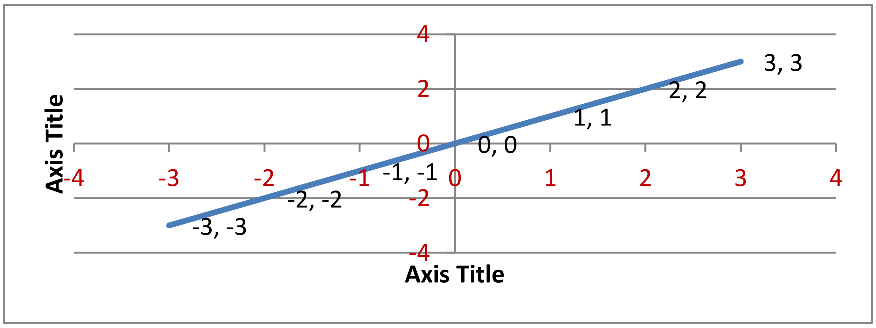
This is also a very simple rule in the sense that the value of y is identical with the value of x saving our time to calculate the value of y.

Dom = R

Range = R

i.e. Domain of the function is same as Range of the function

Graph of Identity Function



Modulus Function

$$y = f(x) = |x| = \begin{Bmatrix} x, x \geq 0 \\ -x, x < 0 \end{Bmatrix}$$

The rule here is: the value of y is always equals to the numerical value of x, not taking in to consideration the sign of x.

Example

Y =f(2)=2

Y=f(0)=0

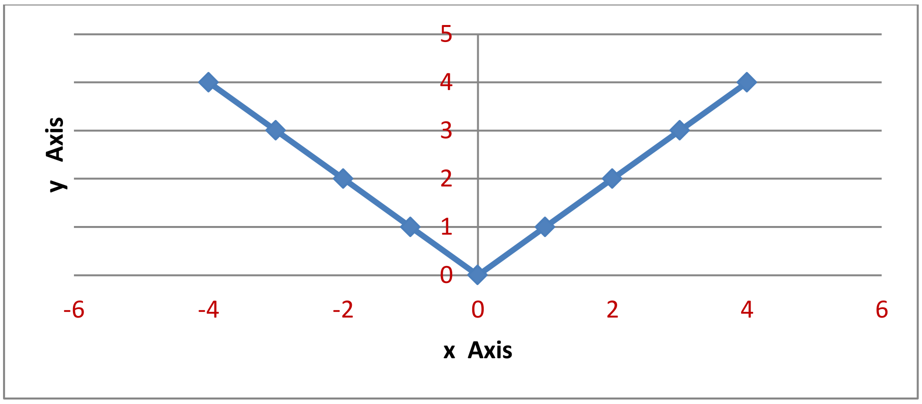
Y=f(-3)=3

This function is usually useful in dealing with values which are always positive for example, length, area etc.

Dom = R

Range = R^+ U $\{0\}$

Graph of Modulus Function



Signum Function

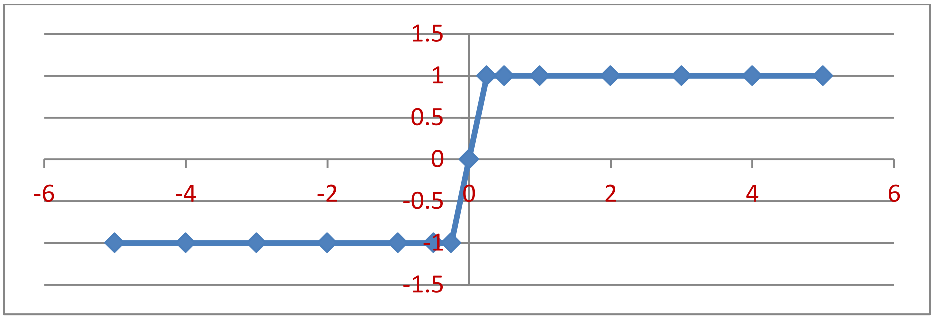
$$y = f(x) = \begin{Bmatrix} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{Bmatrix} = \begin{Bmatrix} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{Bmatrix}$$

This is also a very simple rule in the sense that the value of y is 1 if x is positive , 0 when x=0, and -1 when x is negative.

Dom = R

Range = $\{-1,0,1\}$

Graph of Signum Function



Greatest Integer Function

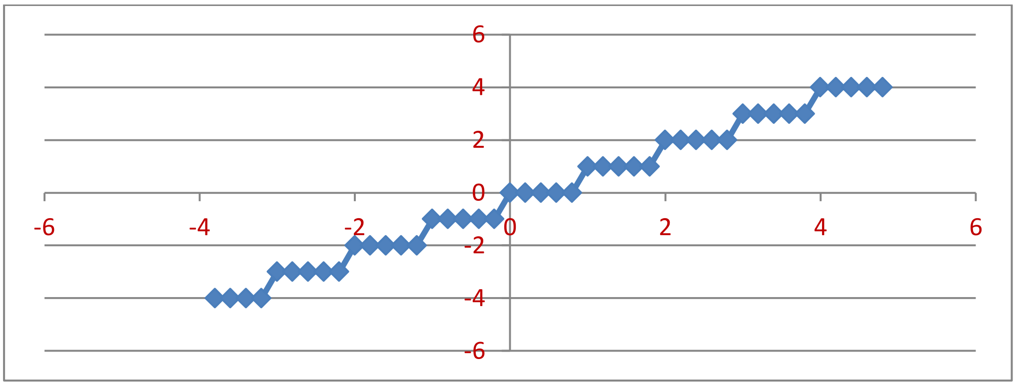
$$y = f(x) = [x] = greatest\ integer \leq x$$

For Example $[0] = 0, [0.2] = 0, [2.5] = 2, [-3.8] = -4\ , etc.$

Dom = R

Range=Z(set of all Integers)

Graph of The function



Exponential Function

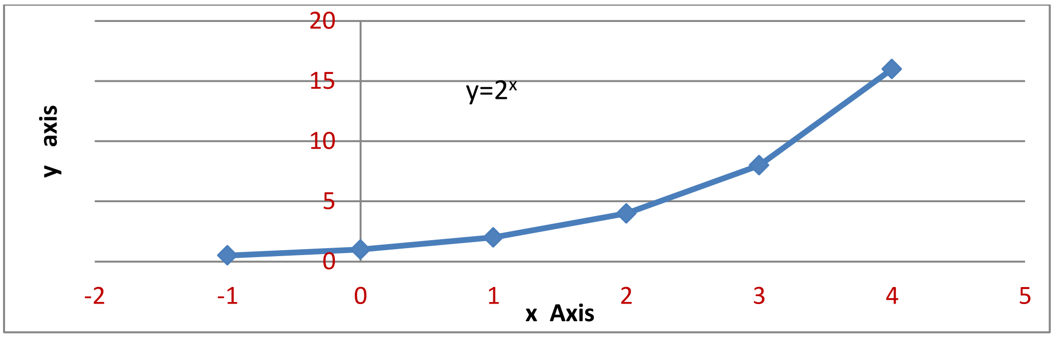
$$y = f(x) = a^x\ where\ a\ > 0$$

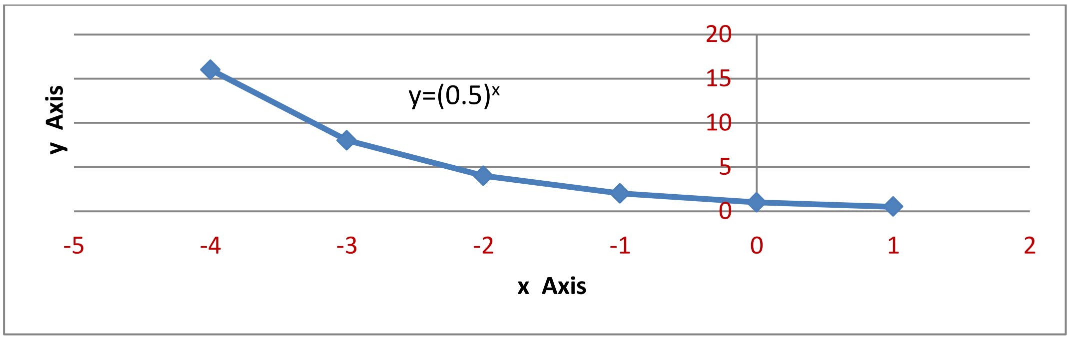
Dom = R

Range= R^+

The specialty of the function is that whatever the value of x, y can never be 0 or negative

Graph of Exponential Function





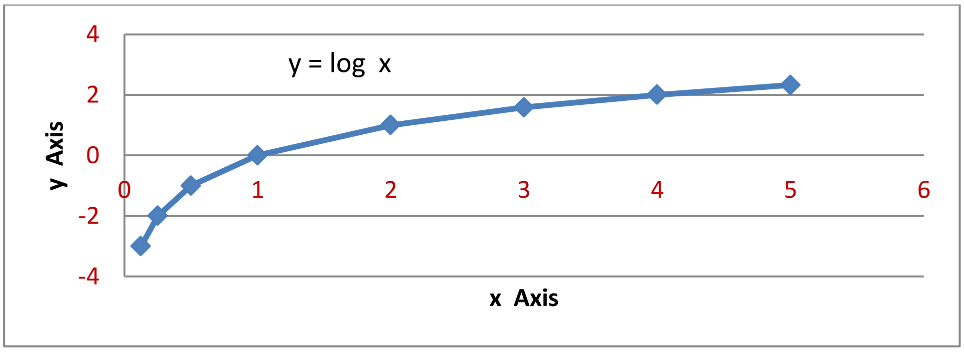
Logarithmic Function

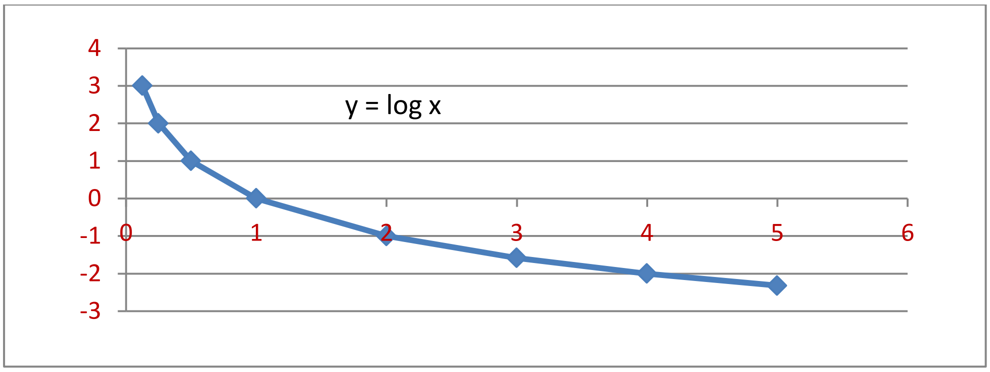
$$y = f(x) = \log_a x$$

Dom = R^+

Range =

Graph of Logarthmic Function





y = log x
4
3
2
1
0
-1
-2
-3
0
1
2
3
4
5
6

LIMIT OF A FUNCTION

Consider the function

$y = 2x + 1$

lets see what happens to value of y as the value of x changes.

Lets take the values of x close to the value of, say, 2. Now when we say value of x close 2. It can be a value like 2.1 or 1.9. in one case it is close to 2 but greater than 2 and in other it is close to 2 but less than 2.Now consider a sequence of such numbers slightly greater than 2 and slightly less than 2 and accordingly calculate the value of y in each case.

Look at the table

x	y=2x+1
1.9	4.8
1.91	4.82
1.92	4.84
1.93	4.86
1.94	4.88
1.95	4.9
1.96	4.92
1.97	4.94
1.98	4.96
1.99	4.98
2.01	5.02
2.02	5.04
2.03	5.06
2.04	5.08
2.05	5.1
2.06	5.12
2.07	5.14
2.08	5.16
2.09	5.18
2.1	5.2

We see in the tabulated value that

as x is approaching the value of 2 from either side, the value of y is approaching the value of 5

in other words we say,

y → 5 (y tends to 5) as x → 2(x tends to 2) or

$$\lim_{x \to 2} \quad y = 5$$

INFINITE LIMIT

As x→ a for some finite value of a, if the value of y is greater than any positive number however large then we say

Y → ∞ (y tends to infinity)

In other words y is said have an infinite limit as x → a. And we write

$$\lim_{x \to a} y = \infty$$

Example

If

$y = \frac{1}{x^2}$,

Then

$$\lim_{x \to 0} y = \infty$$

Since x→0, $x^2 \to 0$ *and* x^2 is positive,

$\frac{1}{x^2}$ becomes very very large and is positive. Therefore the result.

Similarly,

As x→ a for some finite value of a, if the value of y is less than any negative number however large then we say

Y →- ∞ (y tends to minus infinity)

In other words y is said have an infinite limit as x → a. And we write

$$\lim_{x \to a} y = -\infty$$

Example

If

$y = -\frac{1}{x^2}$,

Then

$$\lim_{x \to 0} y = -\infty$$

Since x→0, $x^2 \to 0 \; and \; x^2$ is positive,

$-\frac{1}{x^2}$ becomes very very large and is negative. Therefore the result.

LIMIT AT INFINITY

As x becomes very very large or in other words the value of x is greater than a very large positive number , i.e. x → ∞, if value of y is close to a finite value' a', then we say has a finite limit 'a' at infinity and write

$$\lim_{x \to \infty} y = a$$

Example

Let $y = \frac{1}{x}$

As x→∞ , $\frac{1}{x}$ becomes very very small and approaches the value 0.Therefore we write

$$\lim_{x \to \infty} y = 0$$

similarly

As x becomes very very large with a negative sign or in other words the value of x is less than a very large negative number , i.e. x →- ∞, if value of y is close to a finite value' a', then we say has a finite limit 'a' at infinity and write

$$\lim_{x \to -\infty} y = a$$

Example

Let $y = \frac{1}{x}$

As x→∞ , $\frac{1}{x}$ becomes very very small and approaches the value 0.Therefore we write

$$\lim_{x \to -\infty} y = 0$$

ALGEBRA OF LIMITS

1. Limit of sum of two functions is sum of their individual limits

Let $\lim_{x \to a} f(x) = m \; and \; let \; \lim_{x \to a} g(x) = n$, then

$$\lim_{x \to a} (f(x) + g(x)) = m + n$$

2.Limit of product of two functions is product of the their individual limits

Let $\lim_{x \to a} f(x) = m \; and \; let \; \lim_{x \to a} g(x) = n$, then

$$\lim_{x \to a} (f(x) \times g(x)) = m \times n$$

3. Limit of quotient of two functions is quotient of the their individual limits

Let $\lim_{x \to a} f(x) = m \; and \; let \; \lim_{x \to a} g(x) = n \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{m}{n}$$

SOME STANDARD LIMITS

1. $\lim_{x \to a} \; P(x) = P(a)$ where P(x) is polynomial in x

Example

$$\lim_{x \to 1}(\, 2x^2 + 3x + 1) = 2 \times 1^2 + 3 \times 1 + 1 = 6$$

2.$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ where n is a rational number

Example

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a^{2-1} = 2a$$

3.$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e,$

$\lim_{n \to \infty} \left(1 + \frac{k}{n}\right)^n = e^k,$

4. $\lim_{n \to 0}(1 + n)^{\frac{1}{n}} = e$

$\lim_{n \to 0}(1 + n)^{\frac{k}{n}} = e^k$

5.$\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) \quad = \ln a$

Example

.$\lim_{x \to 0} \left(\frac{2^x - 1}{x}\right) \quad = \ln 2$

6. $\lim_{x \to 0} \frac{log_a(1+x)}{x} = log_a \; e$

Example

$\lim_{x \to 0} \frac{ln(1+x)}{x} = \ln e = 1$

SOME STANDARD TRIGONOMETRIC LIMITS

1$\lim_{x\to 0} \sin x = 0$

2. $\lim_{x\to 0} \cos x = 1$

3. $\lim_{x\to 0} \tan x = 0$

4. $\lim_{x\to 0} \frac{\sin \mathrm{x}}{\mathrm{x}} = 1,$ here $x \to 0\ through\ radian\ values$

5. $\lim_{x\to 0} \frac{\sin \mathrm{x}^0}{\mathrm{x}} = \frac{\pi}{180}$

Example

$$\lim_{x\to 0} \frac{\sin\ \mathrm{mx}}{\sin\ \mathrm{nx}} = \frac{m}{n}$$

Since

$$\lim_{x\to 0} \frac{\sin\ \mathrm{mx}}{\sin\ \mathrm{nx}} = \lim_{x\to 0} \quad \frac{\sin mx}{mx} \times \frac{nx}{\sin nx} \times \frac{m}{n}$$

$$= 1 \times 1 \times \frac{m}{n} = \frac{m}{n}$$

$$\lim_{x\to\infty} \frac{2x^2 + 3x + 5}{3x^2 + 2x + 1} = \lim_{x\to\infty} \frac{2 + \frac{3}{x} + \frac{5}{x^2}}{3 + \frac{2}{x} + \frac{1}{x^2}} = \frac{2}{3}$$

Example

$$\lim_{x\to\infty} \frac{1 + 2 + 3 \dots \dots \dots .. + n}{n^2}$$

$$= \lim_{x\to\infty} \frac{n(n+1)}{2 \times n^2} = \lim_{x\to\infty} \frac{(n^2 + n)}{2 \times n^2} = \lim_{x\to\infty} \frac{(1 + \frac{1}{n})}{2} = \frac{1}{2}$$

Example

$$\lim_{x\to 0}\frac{(1-\cos x)}{x^2} = \lim_{x\to 0} \quad \frac{2sin^2\frac{x}{2}}{x^2} = \frac{sin^2 x}{2\frac{x^2}{4}} = \frac{1}{2}\left(\lim_{x\to 0}\frac{sin\frac{x}{2}}{\frac{x}{2}}\right)\times\left(\lim_{x\to 0}\frac{sin\frac{x}{2}}{\frac{x}{2}}\right)$$

$$= \frac{1}{2}\times 1\times 1 = \frac{1}{2}$$

Existence of Limits

When we say x tends to ‘a’ or write x→a it can happen in two different ways

X can approach ‘a’ through values greater than ‘a’ i.e from right side of ‘a’ on the Number Line

Or

X can approach ‘a’ through values smaller than ‘a’ i.e from left side of ‘a’ on the Number Line

The first case is called the Right Hand Limit and the later case is called the Left Hand Limit.

We, therefore conclude that Limit will exist iff the right Hand Limit and the Left Hand Limit both exist and are EQUAL

Consider the Greatest Integer Function

$$y = f(x) = [x]$$

Consider the limit of this function as $x \to 1$

The right hand limit of this function

$$\lim_{x\to 1+}[x] = 1$$

Since if the value of x is greater than 1 for example 1+h,h>0, then the greatest integer less than equal to 1+h is 1

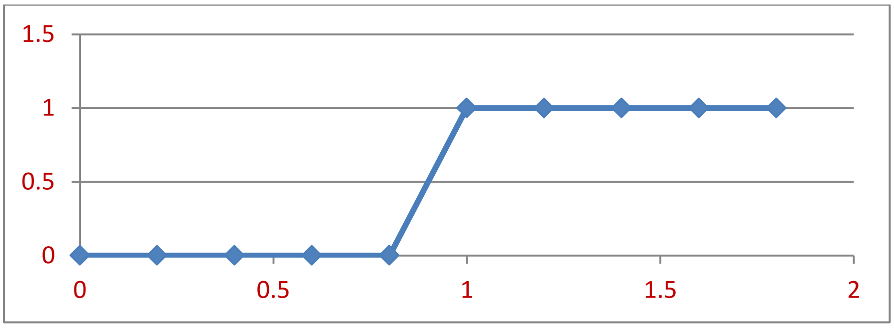
The left hand limit of this function

$$\lim_{x\to 1-}[x] = 0$$

Since if the value of x is less than 1 for example 1-h,h$>$ 0, then the greatest integer less than equal to 1-h is 0

In this case the right hand limit and the left hand limit are not equal

And therefore the limit of this function as $x \to 1$does not exist



For that matter this function does not allow limit as $x \to n$

Since the right hand limit will be always n and the left hand limit will be n-1.

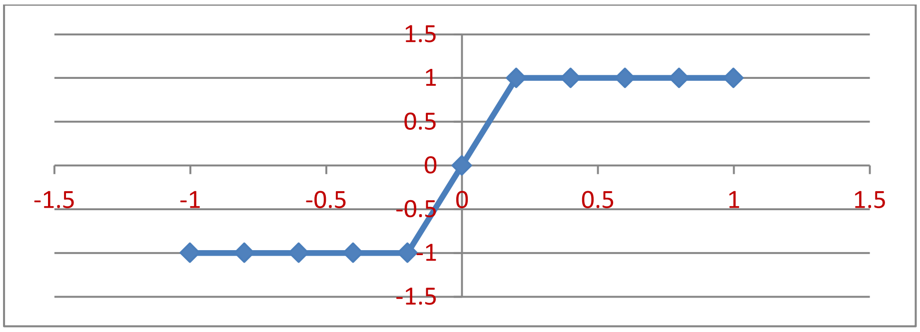
Consider the Signum Function

$$y = f(x) = \begin{Bmatrix} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{Bmatrix} = \begin{Bmatrix} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{Bmatrix}$$

Consider the limit of this function as $x \to 0$

The right hand limit of this function is 1 and the left hand limit of this function is -1 as evident from the definition of the function and concept of right and left hand limits

Therefore this function does not have a limit as $x \to 0$



Continuity of function

A function is continuous at a point 'c' iff its functional value i.e the value of the function at the point 'c' is same as limiting value of the function i.e value of the limit evaluated at the point 'c'

OR

$$\lim_{x \to c} f(x) = f(c)$$

This means that a function is continuous at a point 'c' iff

All the three conditions mentioned below holds good

1.limit of the function as $x \to c$ exists

2.the function has a value at x=c. i.e f(c) does exist

3.the limit of the function is equal to value of the function at the point x=c

Most of the functions we encounter are continuous functions

For example

The physical growth of a child is a continuous function

The distance travelled is a continuous function of time

Continuous functions are easy to handle in the sense that we can predict the value at an latter stage. For example if the education of a child is continuous we can predict what he or she might be reading after say 5 years.

Examples

The constant function is continuous at any point 'c' and hence is continuous everywhere.

$$\lim_{x \to c} f(x) = \lim_{x \to c} K = K \text{ ,} where\ f(x) = \ K\ is\ the\ constant\ function$$

Consider the Function

$$f(x) = \frac{x^2 - 16}{x - 4}$$

This function is not continuous at x=4.Since the function is not defined at x=4

Consider another Function

$$f(x) = [x] \; or \; the \; greatest \; Integer \; Function$$

Consider the point x=2

This function does not have limit x → 2 as the Right Hand limit will be 2 and the Left Hand Limit will be 1.Hence this function is also not continuous at x = 2

Example

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4}, x \neq 4 \\ 8, x = 4 \end{cases}$$

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4}(x + 4) = 8 = f(4)$$

i.e

$$\lim_{x \to 4} f(x) = f(4)$$

This function is therefore continuous at x=4

Limiting value is same as functional value

Consider another Function

$$f(x) = \begin{cases} \left(1 + \dfrac{k}{x}\right)^x, x \neq 0 \\ e^k, x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(1 + \frac{k}{x}\right)^x = \lim_{x \to 0} \left[\left(1 + \frac{k}{x}\right)^{\frac{x}{k}}\right]^k = e^k$$

$$\lim_{x \to 0} f(x) = e^k = f(0)$$

i.e

limit of the function is same as value of the function at the point therefore, the function is continuous at x=0

example

consider the function

$$y = f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider the point x=0

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0 = f(0)$$

Therefore the function is continuous at x=0

As,

$$0 \leq \left|x\sin\frac{1}{x}\right| \leq |x|$$

Taking limit as x→ 0, we can conclude that

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Differentiation

A function f(x) is said to be differentiable at a point x=c iff

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \quad exists$$

In general, a function is differentiable iff

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad exists$$

Once this limit exists, it is called the differential coefficient of f(x) or the derivative of the function f(x) at x=c

Or

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=f'(c)$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$$

Where $f'(c)$ and $f'(x)$ are the differential coefficient or the derivative of the function, the first being defined at x=c

Examples

Consider the function

$$y=f(x)=k \text{ or the constant function}$$

In this case the differential coefficient $f'(x)$is given by

$$f'(x)=\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{k-k}{\delta x}=0$$

Therefore the constant function is differentiable everywhere and the derivative is zero

Consider the function

$$y = f(x) = x^2$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^2 - x^2}{(x + \delta x) - x}$$

$$= 2x$$

Consider the function

$$y = f(x) = \sin x$$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{sin(x + \delta x) \quad - sinx}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2cos\left(\frac{x + \delta x + x}{2}\right) \quad \times sin\left(\frac{x + \delta x - x}{2}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{cos\left(\frac{x + \delta x + x}{2}\right) \quad \times sin\left(\frac{x + \delta x - x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \frac{cos\left(\frac{x + \delta x + x}{2}\right) \quad \times sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$$

$$= \lim_{\delta x \to 0} cos\left(x + \frac{\delta x}{2}\right) \times 1$$

$$= \cos x$$

Therefore

$$y = f(x) = sinx$$

$$\frac{dy}{dx} = cosx$$

Algebra of derivatives

Consider two differentiable functions u(x) and v(x)

Let

$$y = u + v$$

Then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Let

$$y = u \times v$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let

$$y = \frac{u}{v}, \qquad v \neq 0$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example

1

$$y = sinx + x^3$$

$$\frac{dy}{dx} = cosx + 2x$$

2

$$y = x^2 cosx$$

$$\frac{dy}{dx} = x^2(-sinx) + cosx(2x)$$

$$= -x^2 sinx + 2xcosx$$

3

$$y = \frac{sinx}{cosx}$$

$$\frac{dy}{dx} = \frac{cosxcosx - sinx(-sinx)}{(cosx)^2}$$

$$\frac{dy}{dx} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2}$$

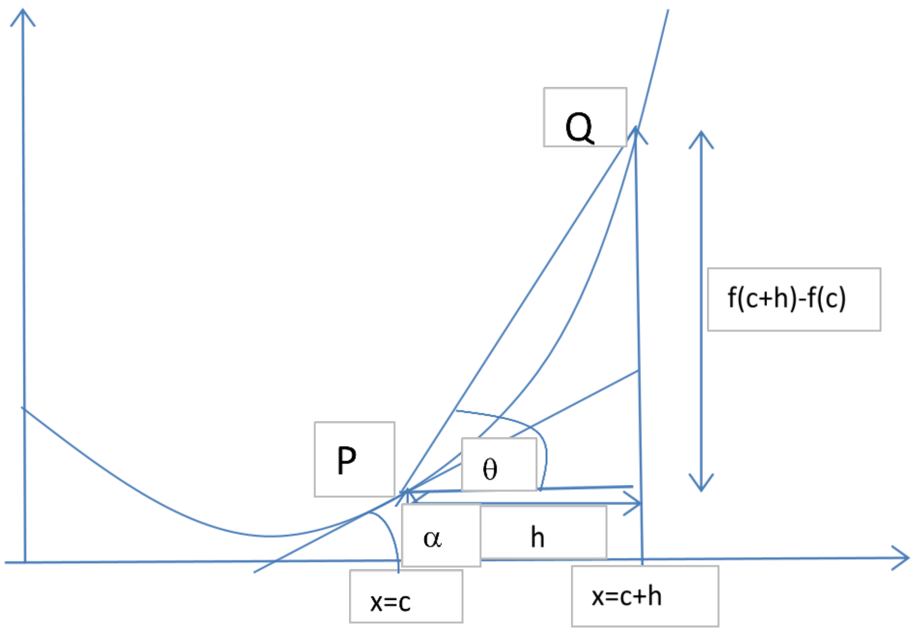
$$\frac{dy}{dx} = \frac{(cosx)^2 + (sinx)^2}{(cosx)^2}$$

$$\frac{dy}{dx} = \frac{1}{(cosx)^2} = (secx)^2$$

Geometrical meaning of $f'(c)$

Consider the graph of a function

$$y = f(x)$$



$$\frac{f(c+h)-f(c)}{h}$$

Represents the ratio of height to base of the angle the line joining the point P(c,f(c)) and Q (c+h,f(c+h))

i.e

$$\frac{f(c+h)-f(c)}{h} = tan\theta$$

Where θ is the angle the line joining the point P and Q makes with the positive direction of x axis.

In the limiting case as h$\rightarrow 0$ i.e as Q$\rightarrow P$ the line PQ becomes the tangent line and the the angle θ *becomes the angle* α which the tangent line makes with the positive direction of x axis

i.e

$$\lim_{h\rightarrow 0}\frac{f(c+h)-f(c)}{h} = f'(c) = tan\alpha = m\ (the\ slope\ of\ the\ tangent)$$

Application to Geometry

To find the equation of the tangent line to the curve y=f(x) at x=x_0

The equation of line passing through the point $(x_0, f(x_0))$ is give by

$$y - f(x_0) = m(x - x_0)$$

Where ‘ m’ is the slope of the tangent line.

As, we have seen

$$m = f'(x_0)$$

The equation is therefore

$$y - f(x_0) = f'(x_0)(x - x_0)$$

In the above example if we take

$f(x) = x^2$ *and the point* $x_0 = 1$

The equation to the tangent at the point is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Or

$$y - 1^2 = 2 \times 1(x - 1)$$

where

$$f(x_0) = x_0^2 = 1^2 \; and \; f'(x_0) = 2 \times x_0 = 2 \times 1$$

i.e

the equation is

$$y - 1 = 2(x - 1)$$

Derivative as rate measurer

Remember the definition

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = f'(c)$$

The quantity

$$\frac{f(c+h) - f(c)}{h}$$

$$measures\ the\ rate\ of\ change\ in\ f(c) wth\ respect\ to\ change\ h\ in\quad 'c'$$

Consider the linear motion of a particle given as

$$s = f(t)$$

Where ‘s’ denotes the distance traversed and ‘t’ denotes the time taken

The ratio

$$\frac{s}{t}$$

Denotes the **average velocity** of the particle

To calculate the local velocity or instantaneous velocity at a point of time t=$t_0$we proceed in the following way

Consider an infinitesimal distance ' $\delta s'$ traversed from time t=t_0 in time ' $\delta t'$

The ratio

$$\frac{\delta s}{\delta t}$$

Still represents a average value of the velocity

The instantaneous velocity at t=t_0 can be calculated by considering the following limit

$$\lim_{\delta t \to 0} \frac{\delta s}{\delta t}$$

or

$$v \ = \frac{ds}{dt}$$

Where 'v' represents the instantaneous ***velocity*** which is defined as rate of change of displacement

Similarly, we can write the mathematical expression for ***acceleration***

As

$$a \ = \frac{dv}{dt}$$

Or the rate of change of velocity

Example

If the motion of a particle is given by

$$s = f(t) = 2t + 5$$

Which is linear in nature, we can calculate velocity at t=3

$$v(t = 3) = \frac{ds}{dt} = 2$$

It is clear that the velocity is independent of time 't'.

i.e

the above motion has constant or uniform velocity.

And, therefore, the acceleration

$$a = \frac{dv}{dt} = 0$$

Or the motion does not produce any acceleration.

Consider another motion of a particle given as

$$s = f(t) = 2t^2 + 3$$

Here the velocity at t=3 can be calculated as

$$v(t = 3) = \frac{ds}{dt} = 4t = 4 \times 3 = 12$$

And the acceleration

$$a = \frac{dv}{dt} = 4$$

Therefore we can say that the motion is said to have constant or uniform acceleration

Derivatives of implicit function

Consider the equation of a circle

$$x^2 + y^2 = r^2$$

This is an implicit function

Lets differentiate this equation with respect x throughout, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Derivative of parametric function

The equation of a circle can also be written as

$$x = rcost$$

$$y = rsint$$

This is called parametric function having parameter ‘t’

In this case

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{rcost}{-rsint} = \frac{x}{-y} = \frac{-x}{y}$$

Derivative of function with respect to another function

Consider the functions

$$y = f(x)$$

$$z = g(x)$$

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)}$$

Example

Let

$$y = sin(x)$$

$$z = x^3$$

$$\frac{dy}{dx} = \frac{f'(x)}{g'(x)} = \frac{cosx}{3x^2}$$

Derivative of composite function

Consider the function

$$y = f(u) \; where \; u = g(x)$$

Then y is called a composite function

In this case

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is called Chain Rule. This can be extended to any number of functions.

Example

1.Let

$$y = sinx^2$$

This can be written as

$$y = sinu$$

And

$$u = x^2$$

Applying chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = cosu \times 2x = 2xcosx^2$$

2.Let

$$y = tane^{x^2}$$

This can be written as

$$y = tanu$$

And

$$u = e^v$$

$$v = x^2$$

Applying chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = sec^2 u \times e^v \times 2x = sec^2 e^{x^2} \times e^{x^2} \times 2x$$

Derivatives of inverse function

$$since \quad \frac{\delta x}{\delta y} = \frac{1}{\frac{\delta y}{\delta x}}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

As $\delta x \to 0, \delta y \; also \to 0$

Which follows from the fact that

$$y = f(x) \; being \; a \; differentiable \; function \; is \; a \; continuous \; function$$

And the condition of continuity guarantees the above fact.

Derivative of inverse trigonometric function

Let

$$y = sin^{-1} x$$

Where $\quad y \epsilon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

This can be written as

$$x = siny$$

$$\frac{dx}{dy} = cosy$$

Or

$$\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\mp\sqrt{(1 - sin^2 y)}} = \frac{1}{\mp\sqrt{(1 - x^2)}} = \frac{1}{\sqrt{(1 - x^2)}}$$

Since cosy is positive in the domain

Let

$$y = cos^{-1}x$$

Where $y\epsilon(0, \pi)$

This can be written as

$$x = cosy$$

$$\frac{dx}{dy} = -siny$$

Or

$$\frac{dy}{dx} = \frac{-1}{siny} = \frac{-1}{\mp\sqrt{(1 - cos^2y)}} = \frac{-1}{\mp\sqrt{(1 - x^2)}} = \frac{-1}{\sqrt{(1 - x^2)}}$$

Since sin y is positive in the domain

Let

$$y = sec^{-1}x$$

Where $y\epsilon\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

This can be written as

$$x = secy$$

$$\frac{dx}{dy} = secy \times tany$$

Or

$$\frac{dy}{dx} = \frac{1}{secy \times tany} = \frac{1}{x\sqrt{(sec^2y - 1)}} = \frac{1}{x(\mp\sqrt{(x^2 - 1)})} = \frac{1}{|x|\sqrt{(1 - x^2)}}$$

Since $secy \times tany$ is positive in the domain

Let

$$y = cosec^{-1}x$$

Where $\quad y\epsilon\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$

This can be written as

$$x = cosecy$$

$$\frac{dx}{dy} = -cosecy \times coty$$

Or

$$\frac{dy}{dx} = \frac{-1}{cosecy \times coty} = \frac{-1}{x\sqrt{(cosec^2y - 1)}} = \frac{-1}{x(\mp\sqrt{(x^2-1)})} = \frac{-1}{|x|\sqrt{(1-x^2)}}$$

Since $cosecy \times coty$ is positive in the domain

Let

$$y = tan^{-1}x$$

This can be written as

$$x = tany$$

$$\frac{dx}{dy} = sec^2y$$

Or

$$\frac{dy}{dx} = \frac{1}{sec^2y} = \frac{1}{1+tan^2y} = \frac{1}{1+x^2}$$

Let

$$y = cot^{-1}x$$

This can be written as

$$x = coty$$

$$\frac{dx}{dy} = -cosec^2y$$

Or

$$\frac{dy}{dx} = \frac{-1}{cosec^2 y} = \frac{-1}{1 + cot^2 y} = \frac{-1}{1 + x^2}$$

Higher order derivatives

Let

$$y = f(x)$$

Is differentiable and also

$$\frac{dy}{dx} = f'(x)$$

Is differentiable. Then we define

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

$$=$$

$$= \lim_{\delta x \to 0} \frac{f'(x + \delta x) - f'(x)}{\delta x}$$

This is the 2^{nd}. Order derivative of the function

Similarly we can define higher order derivatives of the function

Example

Let

$$y = f(x) = x^3 + x^2 + x + 1$$

$$\frac{dy}{dx} = f'(x) = 3x^2 + 2x + 1$$

$$\frac{d^2y}{dx^2} = f''(x) = 6x + 2$$

Consider the Function

$$y = f(x) = Acosx + Bsinx$$

Here

$$\frac{dy}{dx} = f'(x) = -Asinx + Bcosx$$

$$\frac{d^2y}{dx^2} = f''(x) = -Acosx - Bsinx = -y$$

i.e in this case

$$\frac{d^2y}{dx^2} + y = 0$$

Monotonic Function

Increasing function

Consider a function

$$y = f(x)$$

If $x_2 > x_1 \; implies \; f(x_2) > f(x_1)$

Then the function is increasing

Example

$$y = f(x) = x + 1$$

$$f(2) = 2 + 1 = 3$$

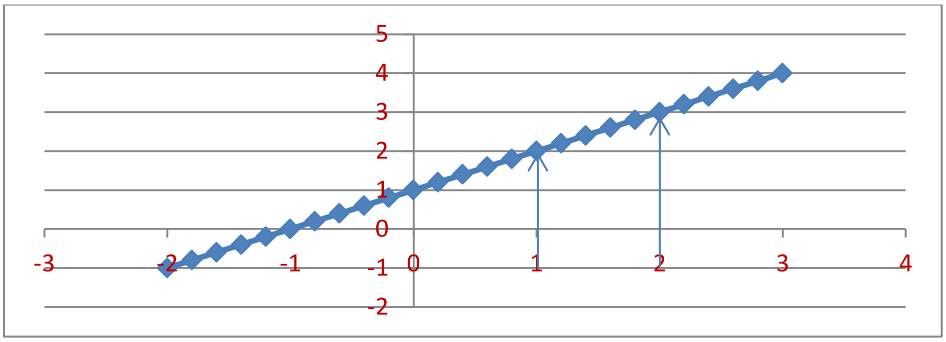
$$f(1) = 1 + 1 = 2$$

Or

$$f(2) > f(1)$$

Therefore the function is increasing

Graph of the function



Decreasing function

Consider a function

$$y = f(x)$$

If $x_2 > x_1 \ \ implies \ f(x_2) < f(x_1)$

Then the function is decreasing

Consider the function

$$y = f(x) = \frac{1}{x}$$

$$f(2) = \frac{1}{2}$$

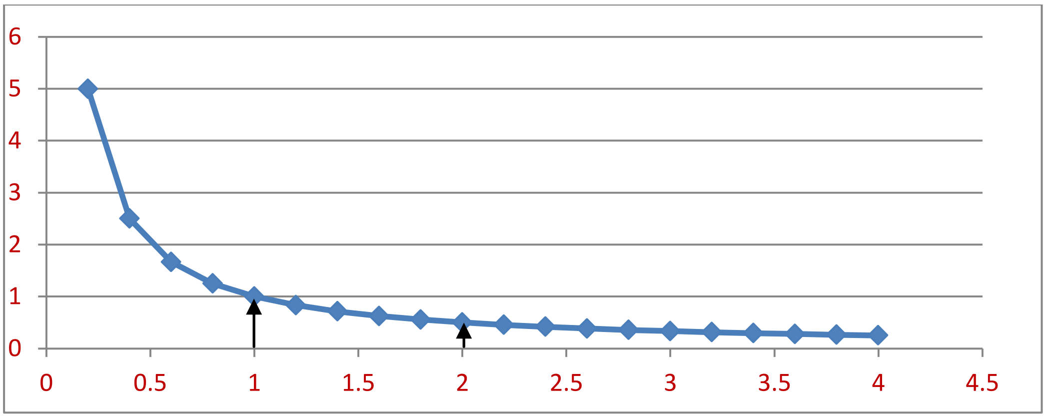
$$f(1) = \frac{1}{1} = 1$$

Or

$$f(2) < f(1)$$

Therefore the function is decreasing

Graph of the function



A function either increasing or decreasing is called monotonic.

Derivative of Increasing Function

If f(x) is increasing, then

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} > 0$$

i.e

for increasing function the derivative is always positive

Derivative of Decreasing Function

If f(x) is decreasing, then

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} < 0$$

i.e

for decreasing function the derivative is always negative

example

let

$$y = f(x) = x + 1$$

$$\frac{dy}{dx} = f'(x) = 1 > 0$$

Therefore the function is increasing

Let

$$y = f(x) = \frac{1}{x}$$

$$\frac{dy}{dx} = f'(x) = \frac{-1}{x^2} < 0$$

Therefore the function is decreasing

Let

$$y = f(x) = x^2$$

$$\frac{dy}{dx} = f'(x) = 2x$$

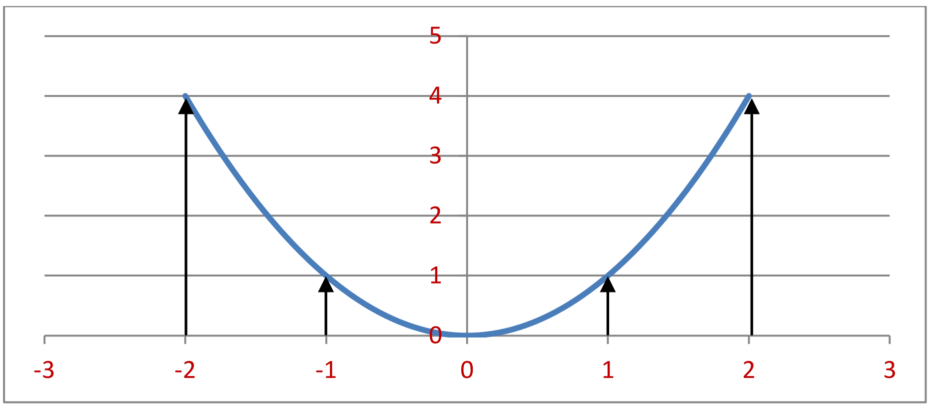
$$> 0 \; for \; x > 0$$

$$< 0 \; for \; x < 0$$

Therefore the function is increasing for $x > 0$ and decreasing for $x < 0$

Graph of the function

$$y = f(x) = x^2$$



MAXIMA AND MINIMA OF A FUNCTION

Consider a function

$$y = f(x)$$

Consider the point x=c

If at this point

$$f(c) > f(c+h), where\ |h| < \delta$$

Then f(c) is called local maximum or simply a maximum of the function

If at this point

$$f(c) < f(c+h), where\ |h| < \delta$$

Then f(c) is called a local minimum or simply a minimum

A function can have several local maximum values and several local minimum values in its domain and it is possible that a local minimum can be larger than a local maximum.

If f (c) is a local maximum then the graph of the function in the domain

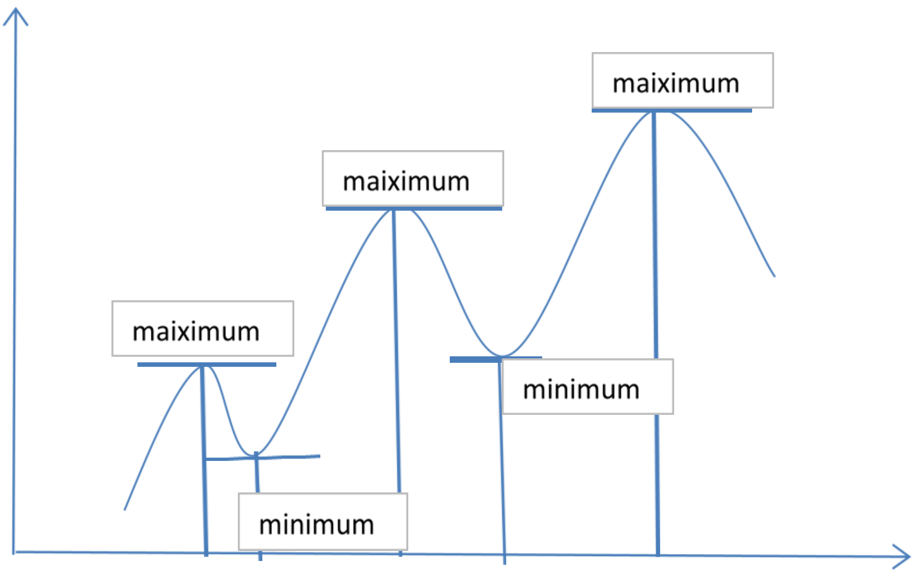
$$(c-\delta, c+\delta)$$

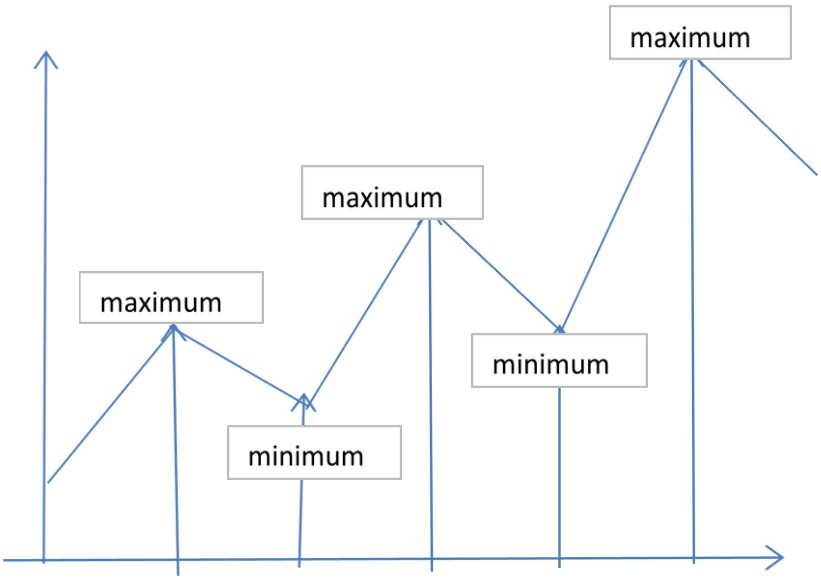
Will be concave downwards

If f (c) is a local minimum then the graph of the function in the domain

$$(c-\delta, c+\delta)$$

Will be concave upwards

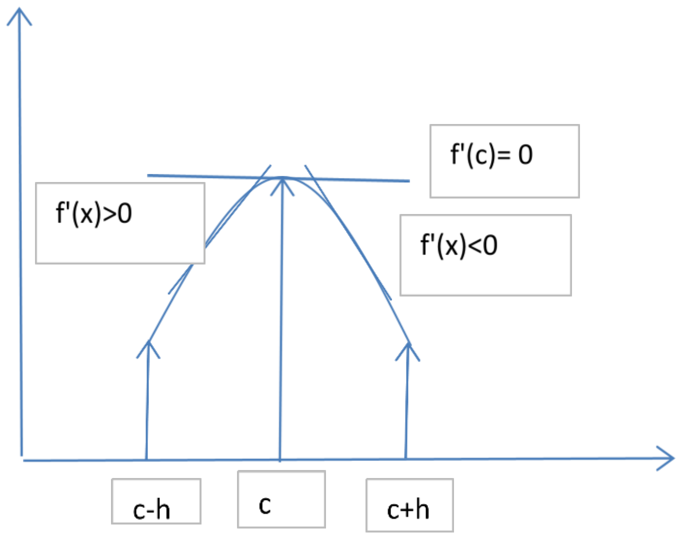


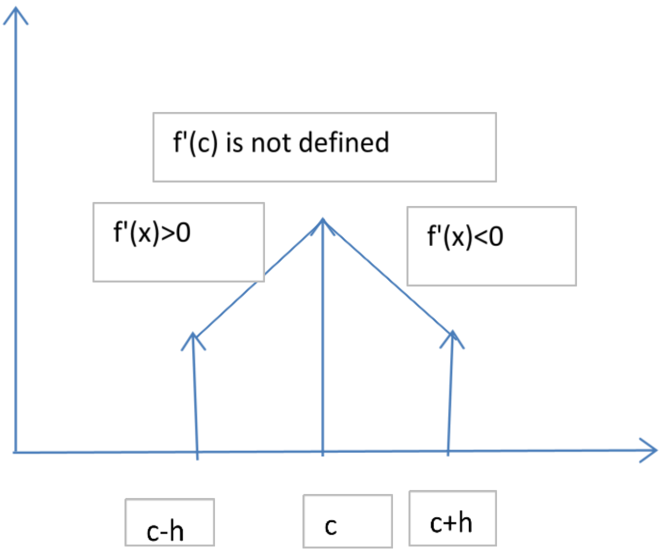


Maximum Case

In other words at a point of local maximum the function is increasing on the left of the point and decreasing on the right of the point

Therefore the derivative of the function changes sign from positive to negative as it passes through x=c



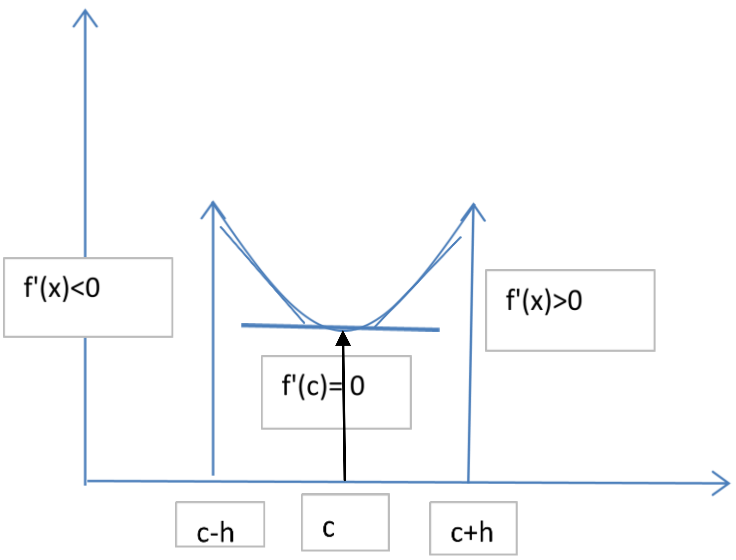


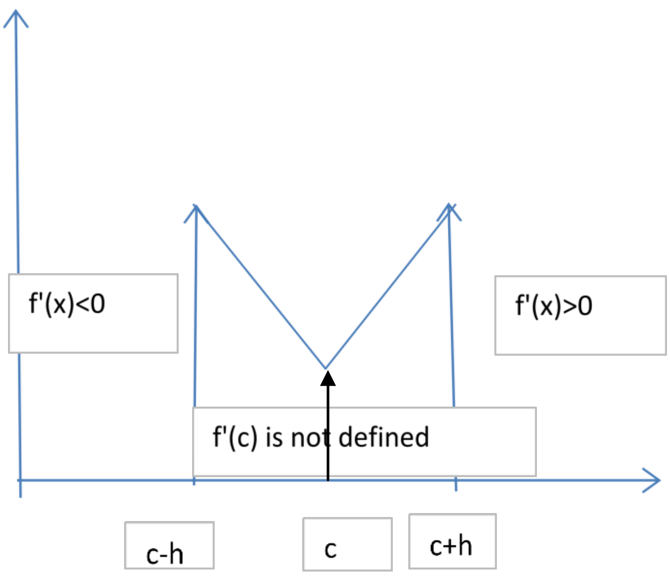
Therefore we conclude that the derivative of the function is a decreasing function and as such its derivative i.e the second order derivative is negative

Minimum Case

At a point of local minimum the function is decreasing on the left of the point and increasing on the right of the point

Therefore the derivative of the function changes sign from negative to positive as it passes through x=c





Therefore we conclude that the derivative of the function is a increasing function and as such its derivative i.e the second order derivative is positive

In either maximum or minimum case the 1st. derivative of the function is zero or is not defined at the point of maximum or minimum

The point x=c where the derivative vanishes or does not exist at all is called a critical point or turning point or stationary point.

A function can have neither a maximum nor a minimum value

Example

Consider the function

$$y = f(x) = x^3$$

Here

$$\frac{dy}{dx} = 3x^2$$

This vanishes at x=0

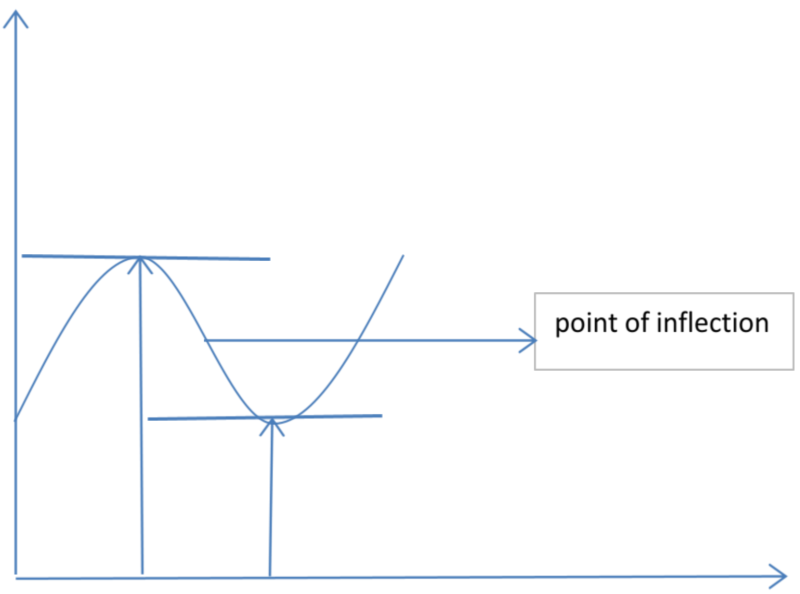
And

$$\frac{d^2y}{dx^2} = 6x$$

Which also vanishes

Therefore we may conclude that the function does not have maximum neither minimum value

Point of inflexion



If a curve is changing its nature from concave downwards to concave upwards as shown in the figure or vice versa, then at the point where this change occurs is called the point of inflexion. In other words on one side of the point of inflexion the curve is concave downward and on the other side the curve is concave upward or vice versa

In the above figure,

On the left side of point of inflexion a maximum value occurs and to the right side of point of inflexion a minimum value occurs.

In other words, remembering the condition of maximum and minimum, we can say,

The 2nd order derivative changes its sign from negative to positive as in the case given in the figure or vice versa.

In other words the point of inflexion is the point of either maximum or minimum of the 1st derivative of the function

Hence at the point of inflexion the 2nd. order derivative vanishes or is not defined and the 2nd. order derivative changes its sign as it passes through the point of inflexion i.e at the point of inflexion

1. $\frac{d^2y}{dx^2} = f''(x) = 0$ or is not defined
2. The 2nd.order derivative changes sign as it passes through the point

Example

Consider the function we discussed earleier

$$y = f(x) = x^3$$

Here

$$\frac{dy}{dx} = 3x^2$$

This vanishes at x=0

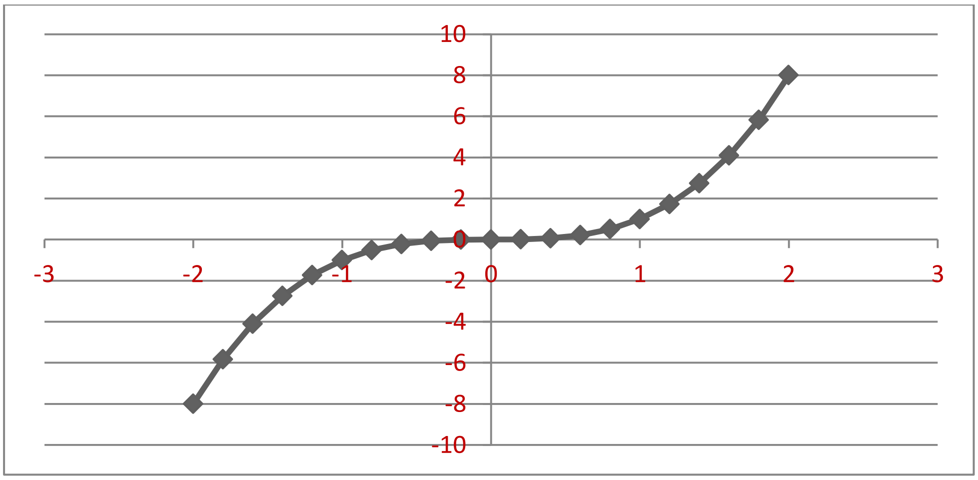
And

$$\frac{d^2y}{dx^2} = 6x$$

Which also vanishes at x=0

But

$$\frac{d^3y}{dx^3} = 6 \neq 0$$



Therefore we conclude that

X=0 is a point of inflexion for the curve

Working procedure to find the maxima and minima

1.Given any function, equate the first derivative to zero to find the turning points or critical points

2.Test the sign of the second derivative at these points. If the sign is negative it is appoint of maximum value. If the sign is positive it is a point of minimum value.

3. then calculate the maximum value/minimum value of the function by taking the value of x as the point

Example

If the sum of two numbers is 10, find the numbers when their product is maximum

Solution

Let the numbers be x and 10-x

Let

$$y = f(x) = x(10 - x)$$

$$= 10x - x^2$$

$$\frac{dy}{dx} = 10 - 2x = 0$$

$$x = 5$$

$$\frac{d^2y}{dx^2} = -2 < 0$$

Therefore the function which is the product of the numbers maximum if the numbers are equal i.e 5 and 5.

EXAMPLE

Investigate the extreme values of the function

$$f(x) = x^4 - 2x^2 + 3$$

The critical points are roots of the equation

$$f'(x) = 4x^3 - 4x \quad = 0$$

Or

$$f'(x) = 4x(x^2 - 1) = 0$$

Or

$$x = 0, x = 1, x = -1$$

Lets check the sign of the 2nd. Derivative at these points

Now,

$$f''(x) = 4(3x^2 - 1)$$

$$f''(0) = 4(-1) = -4 < 0$$

Therefore x=0 is a point of maximum value.

The maximum value of the function is given as

$$f(x)_{max} = f(0) = 3$$

Now

$$f''(1) = 4(3 - 1) = 8 > 0$$

Therefore x = 1 is a point of minimum value.

The minimum value of the function is given as

$$f(x)_{min} = f(1) = 1 - 2 + 3 = 2$$

Now

$$f''(-1) = 4(3 - 1) = 8 > 0$$

Therefore x = -1 is a point of minimum value.

The minimum value of the function is given as

$$f(x)_{min} = f(-1) = 1 - 2 + 3 = 2$$

INTEGRATION
&
DIFFERENTIAL EQUATIONS

INTEGRATION AS INVERSE PROCESS OF DIFFERENTIATION

Integration is the process of inverse differentiation .The branch of calculus which studies about Integration and its applications is called Integral Calculus.

Let $F(x)$ and $f(x)$ be two real valued functions of x such that,

$$\frac{d}{dx}F(x) = f(x)$$

Then, $F(x)$ is said to be an anti-derivative (or integral) of $f(x)$.
Symbolically we write$\int f(x)\, dx = F(x)$.

The symbol $\int$ denotes the operation of integration and called the integral sign.
$'dx'$ denotes the fact that the Integration is to be performed with respect to x .The function $f(x)$ is called the Integrand.

INDEFINITE INTEGRAL

Let $F(x)$be an anti-derivative of $f(x)$.
Then, for any constant 'C',

$$\frac{d}{dx}\{F(x) + C\} = \frac{d}{dx}F(x) = f(x)$$

So,$F(x) + C$ is also an anti-derivative of $f(x)$, where C is any arbitrary constant. Then, $F(x) + C$ denotes the family of all anti-derivatives of $f(x)$, where C is an indefinite constant.
Therefore, $F(x) + C$ is called the Indefinite Integral of $f(x)$.
Symbolically we write

$$\int f(x)\, dx = F(x) + C,$$

Where the constant C is called the constant of integration. The function $f(x)$ is called the Integrand.

Example **:**-Evaluate $\int \cos x \; dx$.
Solution:-We know that

$$\frac{d}{dx}\sin x = \cos x$$

So,

$$\int \cos x \; dx = \sin x + C$$

ALGEBRA OF INTEGRALS

1.$\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$

2. $\int k\, f(x)dx = k \int f(x)dx,$ for any constant k.

3.$\int [a\, f(x) + b\, g(x)]\, dx = a \int f(x)dx + b \int g(x)dx,$
for any constant a & b

INTEGRATION OF STANDARD FUNCTIONS

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C \ , (n \neq -1)$

2. $\int \frac{1}{x} dx = \ln|x| + C$

3. $\int cosx\ dx = \sin x + C$

4. $\int \sin x\ dx = -\cos x + C$

5. $\int sec^2 x\ dx = \tan x + C$

6. $\int cosec^2 x\ dx = -\cot x + C$

7. $\int \sec x \tan x\ dx = \sec x + C$

8. $\int \text{cosec}\, x \cot x\ dx = -\text{cosec}\, x + C$

9. $\int e^x dx = e^x + C$

10. $\int a^x dx = \frac{a^x}{\ln a} + C \ , (a > 0)$

11. $\int \tan x\ dx = \ln|\sec x| + C = -\ln|\cos x| + C$

12. $\int \cot x\ dx = \ln|\sin x| + C = -\ln|\text{cosec}\, x| + C$

13. $\int \sec x\ dx = \ln|\sec x + \tan x| + C$

14. $\int \text{cosec}\, x\ dx = \ln|\text{cosec}\, x - \cot x| + C$

15. $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + C$

16. $\int \frac{1}{1+x^2} dx = tan^{-1}x + C$

17. $\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}x + C$

18. $\int \frac{1}{\sqrt{x^2+1}} dx = \ln\left|x + \sqrt{x^2+1}\right| + C$

19. $\int \frac{1}{\sqrt{x^2-1}} dx = \ln\left|x + \sqrt{x^2-1}\right| + C$

Example:- Evaluate $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

Solution:- $\int \frac{a^2 sin^2 x + b^2 cos^2 x}{sin^2 2x} dx$

$= \int \frac{a^2 sin^2 x + b^2 cos^2 x}{4 sin^2 x . cos^2 x} dx$

$= \frac{a^2}{4} \int \frac{1}{cos^2 x} dx + \frac{b^2}{4} \int \frac{1}{sin^2 x} dx$

$= \frac{a^2}{4} \int sec^2 x\ dx + \frac{b^2}{4} \int cosec^2 x\ dx$

$= \frac{1}{4}[a^2 \tan x - b^2 \cot x] + C$

INTEGRATION BY SUBSTITUTION

When the integrand is not in a standard form, it can sometimes be transformed to integrable form by a suitable substitution.

The integral $\int f\{g(x)\}g'(x)dx$ can be converted to
$\int f(t)dt$ by substituting $g(x)$by t.

So that, if $\int f(t)dt = F(t) + C$,then

$\int f\{g(x)\}g'(x)dx = F\{g(x)\} + C$.

This is a direct consequence of CHAIN RULE.
For,

$$\frac{d}{dx}[F\{g(x)\} + C] = \frac{d}{dt}[F(t) + C].\frac{dt}{dx} = f(t).\frac{dt}{dx} = f\{g(x)\}g'(x)$$

There is no fixed formula for substitution.

Example:- Evaluate $\int \cos(2 - 7x)\, dx$
Solution:- Put $t = 2 - 7x$
So that $\frac{dt}{dx} = -7 \Longrightarrow dt = -7dx$

$$\therefore \quad \int \cos(2 - 7x)\, dx = \frac{-1}{7}\int \cos t\, dt$$
$$= \frac{-1}{7}\sin t + C$$
$$= \frac{-1}{7}\sin(2 - 7x) + C$$

INTEGRATION BY DECOMPOSITION OF INTEGRAND

If the integrand is of the form $\sin mx \cdot \cos nx$, $\cos mx \cdot \cos nx$ or $\sin mx \cdot \sin nx$, then we can decompose it as follows;

1. $\sin mx \cdot \cos nx = \frac{1}{2} \cdot 2\sin mx \cdot \cos nx = \frac{1}{2}[\sin(m + n)x + \sin(m - n)x]$
2. $\cos mx \cdot \cos nx = \frac{1}{2}[\cos(m - n)x + \cos(m + n)x]$
3. $\sin mx \cdot \sin nx = \frac{1}{2}[\cos(m - n)x - \cos(m + n)x]$

Similarly, in many cases the integrand can be decomposed into simpler form, which can be easily integrated.
Example:-Integrate $\int \sin 5x \cdot \cos 2x\, dx$

Solution:- $\int \sin 5x \cdot \cos 2x\, dx$

$$= \frac{1}{2}\int [\sin(5 + 2)x + \sin(5 - 2)x]\, dx$$
$$= \frac{1}{2}\int (\sin 7x + \sin 3x)\, dx$$
$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x - \frac{1}{3}\cos 3x\right] + C$$
$$= -\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + C$$

Example:-Integrate $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx$

Solution:- $\int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx = \int \frac{2\sin 5x \cos x}{2\cos 5x \cos x}dx$

$$= \int \frac{\sin 5x}{\cos 5x}dx$$

Put t = cos 5x, so that $\frac{dt}{dx} = -5\, sin5x \implies dt = -5sin5x.dx$

$$\therefore \int \frac{\sin 6x + \sin 4x}{\cos 6x + \cos 4x}dx = -\frac{1}{5}\int \frac{dt}{t} = -\frac{1}{5}ln|t| + C$$

$$= -\frac{1}{5}\ln|\cos 5x| + C$$
$$= \frac{1}{5}\ln|\sec 5x| + C$$

<u>INTEGRATION BY PARTS</u>

This rule is used to integrate the product of two functions.
If u and v are two differentiable functions of x, then according to this rule have;

$$\int uv\, dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx\right] dx$$

In words, Integral of the product of two functions

$$= \boldsymbol{first\ function} \times (\boldsymbol{Integral\ of\ second\ function}) - \boldsymbol{Integral\ of}(\boldsymbol{derivative\ of\ first} \times \boldsymbol{Integral\ of\ second})$$

The rule has been applied with a proper choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' functions .Usually from among exponential function(**E**),trigonometric function(**T**),algebraic function(**A**),Logarithmic function(**L**) and inverse trigonometric function(**I**),the choice of '$\boldsymbol{First}$' and '$\boldsymbol{Second}$' function is made in the order of **<u>ILATE</u>.**

<u>Example</u>:-Evaluate$\int x \sin x\, dx$

<u>Solution</u>:-$\int x \sin x\, dx$

$$= x \int \sin x\, dx - \int \left[\frac{dx}{dx}. \int \sin x\, dx\right] dx$$
$$= -x \cos x + \int \cos x\ dx$$
$$= \sin x - x\cos x + C$$

<u>Example</u>:-Evaluate$\int e^x \cos 2x\, dx$

<u>Solution</u>:- $\int e^x \cos 2x\, dx = e^x \cos 2x - \int e^x(-2\sin 2x)\, dx$

$$= e^x \cos 2x + 2\int e^x \sin 2x\, dx$$
$$= e^x \cos 2x + 2\left[e^x \sin 2x - 2\int e^x \cos 2x\, dx\right]$$
$$= e^x \cos 2x + 2\, e^x \sin 2x - 4\int e^x \cos 2x\, dx + K$$

So,$5\int e^x \cos 2x = e^x[\cos 2x + 2\sin 2x] + K$

$\therefore \int e^x \cos 2x\, dx = \frac{e^x}{5}[\cos 2x + 2\sin 2x] + C$ (where $= K/2$)

<u>INTEGRATION BY TRIGONOMETRIC SUBSTITUTION</u>

The irrational forms $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$ can be simplified to radical free functions as integrand by putting $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ respectively.
The substitution $x = a\tan\theta$ can be used in case of presence of $x^2 + a^2$ in the integrand, particularly when it is present in the denominator.

<u>ESTABLISHMENT OF STANDARD FORMULAE</u>

1. $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

Solutions:

1. Let $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$ and $\theta = sin^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2sin^2\theta}} = \int \frac{a\cos\theta}{a\cos\theta} d\theta = \int d\theta = \theta + C = sin^{-1}\frac{x}{a} + C$

2. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$ and $\theta = tan^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x^2+a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta + 1)} = \int \frac{asec^2\theta}{a^2sec^2\theta} d\theta = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$
$= \frac{1}{a} tan^{-1}\frac{x}{a} + C$

3. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\, d\theta$ and $\theta = sec^{-1}\frac{x}{a}$

$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{a\sec\theta\sqrt{a^2sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\sec\theta\, a\tan\theta} d\theta = \frac{1}{a}\int d\theta$
$= \frac{1}{a}\theta + C = \frac{1}{a} sec^{-1}\frac{x}{a} + C$

4. Let $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$.

$\therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2tan^2\theta + a^2}} = \int \frac{asec^2\theta}{a\sec\theta} d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + K$

$=\ln\left|\sqrt{tan^2\theta + 1} + \tan\theta\right| + K = \ln\left|\sqrt{\frac{x^2}{a^2} + 1} + \frac{x}{a}\right| + K$

$=\ln\left|\frac{x + \sqrt{x^2+a^2}}{a}\right| + K$

$=\ln\left|x + \sqrt{x^2+a^2}\right| + K - \ln|a|$

$=\ln\left|x + \sqrt{x^2+a^2}\right| + C$ (Where $C = K - \ln|a|$)

5. Let $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\, d\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2sec^2\theta - a^2}} = \int \frac{a\sec\theta\tan\theta}{a\tan\theta} d\theta = \int \sec\theta\, d\theta$

$=\ln|\sec\theta + \tan\theta| + K = \ln\left|\sec\theta + \sqrt{sec^2\theta - 1}\right| + K$

$= \ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + K$

$=\ln\left|\frac{x + \sqrt{x^2-a^2}}{a}\right| + K$

$=\ln\left|x + \sqrt{x^2-a^2}\right| + K - \ln|a|$

$=\ln\left|x + \sqrt{x^2-a^2}\right| + C$ (Where $C = K - \ln|a|$)

SOME SPECIAL FORMULAE

1. $\int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$

2. $\int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2+a^2}\right| + C$

3. $\int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2-a^2}\right| + C$

Solutions:

1. $\int \sqrt{a^2-x^2}dx = \int 1 \cdot \sqrt{a^2-x^2}dx$

$= x\sqrt{a^2-x^2} - \int x\left(\frac{-2x}{2\sqrt{a^2-x^2}}\right)dx$

$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2-x^2} + \int \frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}} - \int \sqrt{a^2-x^2}dx$

$\therefore \quad 2\int \sqrt{a^2-x^2}dx = x\sqrt{a^2-x^2} + a^2\int \frac{dx}{\sqrt{a^2-x^2}}$

$= x\sqrt{a^2-x^2} + a^2 sin^{-1}\frac{x}{a} + K$

$\therefore \quad \int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\frac{x}{a} + C$ (Where $C = \frac{K}{2}$)

2. $\int \sqrt{x^2+a^2}dx = \int 1 \cdot \sqrt{x^2+a^2}dx$

$= x\sqrt{x^2+a^2} - \int x\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)dx$

$= x\sqrt{x^2+a^2} - \int \frac{x^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2)-a^2}{\sqrt{x^2+a^2}}dx$

$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}dx + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$

$\therefore \quad 2\int \sqrt{x^2+a^2}dx = x\sqrt{x^2+a^2} + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$

So, $2\int \sqrt{x^2+a^2}dx = x\sqrt{x^2+a^2} + a^2 \ln\left|x+\sqrt{x^2+a^2}\right| + K$

$\therefore \quad \int \sqrt{x^2+a^2}dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + C$

(Where $C = \frac{K}{2}$)

3. $\int \sqrt{x^2-a^2}dx = \int 1 \cdot \sqrt{x^2-a^2}dx$

$= x\sqrt{x^2-a^2} - \int x\left(\frac{2x}{2\sqrt{x^2-a^2}}\right)dx$

$= x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}dx$

$= x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}dx$

$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}dx + a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

$\therefore \quad 2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$

So, $2\int \sqrt{x^2-a^2}dx = x\sqrt{x^2-a^2} - a^2 \ln\left|x+\sqrt{x^2-a^2}\right| + K$

$\therefore \quad \int \sqrt{x^2-a^2}dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$

(Where $C = \frac{K}{2}$)

METHOD OF INTEGRATION BY PARTIAL FRACTIONS

If the integrand is a proper fraction $\frac{P(x)}{Q(x)}$, then it can be decomposed into simpler partial fractions and each partial fraction can be integrated separately by the methods outlined earlier.

SOME SPECIAL FORMULAE

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$

Solutions:

1. We have, $\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$

$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + C$

$\therefore \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

2. We have, $\frac{1}{a^2-x^2} = \frac{1}{(a+x)(a-x)}$

$= \frac{1}{2a}\left(\frac{1}{a+x} + \frac{1}{a-x}\right)$

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\int\left(\frac{1}{a+x} + \frac{1}{a-x}\right) dx$

$= \frac{1}{2a}[\ln|a+x| - \ln|a-x|] + C$

$\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$

Example:- Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)} dx$

Solution:- Let $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$ ---------------(1)

Multiplying both sides of (1) by $(x-1)^2(x+3)$,

$\Rightarrow x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$ ---------------(2)

Putting $x = 1$ in (2),we get,$B = \frac{1}{2}$

Putting $x = -3$ in (2),we get,$10 = 16C \Rightarrow C = \frac{5}{8}$

Equating the co-efficients of x^2 on both sides of (2), we get

$1 = A + C \Rightarrow A = 1 - \frac{5}{8} = \frac{3}{8}$

Substituting the values of A,B &C in (1) ,we get

$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{3}{8}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x-1)^2} + \frac{5}{8}\cdot\frac{1}{x+3}$

$\therefore \int \frac{x^2+1}{(x-1)^2(x+3)} dx$

$= \frac{3}{8}\int\frac{dx}{x-1} + \frac{1}{2}\int\frac{dx}{(x-1)^2} + \frac{5}{8}\int\frac{dx}{x+3}$

$= \frac{3}{8}\ln|x-1| + \frac{5}{8}\ln|x+3| - \frac{1}{2(x-1)} + C$

Example:- Evaluate $\int \frac{x}{(x-1)(x^2+4)} dx$

Solution:- Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ ---------------(1)
Multiplying both sides of (1) by $(x-1)(x^2+4)$,we get
$x = A(x^2+4) + (Bx+C)(x-1)$--------------(2)
Putting $x = 1$ in (2), we get,$A = \frac{1}{5}$
Putting $x = 0$ in (2), we get,$0 = 4A - C \Longrightarrow C = 4A \Longrightarrow C = \frac{4}{5}$
Equating the co-efficients of x^2 on both sides of (2), we get
$0 = A + B \Rightarrow B = -\frac{1}{5}$
Substituting the values of A, B and C in (1) we get

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$\therefore \quad \int \frac{x}{(x-1)(x^2+4)} dx = \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x-4}{x^2+4} dx$
$= \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$
$= \frac{1}{5}\int \frac{dx}{x-1} + \frac{1}{10}\int \frac{2xdx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$
$=\frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| + \frac{2}{5}tan^{-1}\left(\frac{x}{2}\right) + C$

Example:- Evaluate $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Solution:- Let $x^2 = y$ $Then \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$

Let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ ---------------(1)
Multiplying both sides of (1) by $(y+1)(y+4)$,we get
$y = A(y+4) + B(y+1)$-----------(2)
Putting $y = -1$ and $y = -4$ successively in (2),we get,$A = -\frac{1}{3}$ and$B = \frac{4}{3}$
Substituting the values of A and B in (1),we get
$\frac{\square}{(\square+1)(\square+4)} = -\frac{1}{3(\square+1)} + \frac{4}{3(\square+4)}$
Replacing $\square$ by $\square^2$, we obtain
$\frac{\square^2}{(\square^2+1)(\square^2+4)} = -\frac{1}{3(\square^2+1)} + \frac{4}{3(\square^2+4)}$
$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{-1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$
$= -\frac{1}{3}tan^{-1}x + \frac{2}{3}tan^{-1}\left(\frac{x}{2}\right) + C$

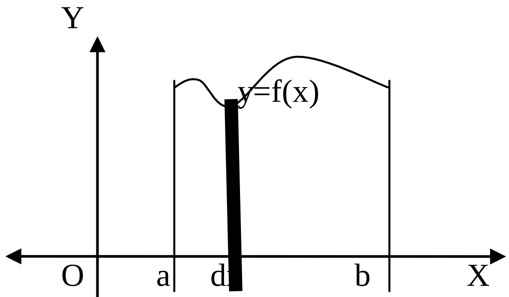
DEFINITE INTEGRAL

If $f(x)$ is a continuous function defined in the interval [a,b] and$F(x)$ is an anti-derivative of $f(x)$ $i.e.,\frac{dF(x)}{dx} = f(x)$,then the define integral of $f(x)$ over [a,b] is denoted by

$\int_a^b f(x)dx$ and is equal to $F(b) - F(a)$

$i.e., \int_a^b f(x)dx = F(b) - F(a)$

The constants a and b are called the limits of integration. ‘a’ is called the lower limit and ‘b’ the upper limit of integration.The interval [a,b] is called the interval of integration.



Geometrically, the definite integral $\int_a^b f(x)dx$ is the AREA of the region bounded by the curve $y = f(x)$ and the lines $x = a, x = b$ and x-axis.

EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)dx$ of a continuous function $f(x)$ defined on [a, b], we use the following steps.

Step-1:-Find the indefinite integral $\int f(x)dx$

Let $\int f(x)dx = F(x)$

Step-2:-Then, we have

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$$

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz$

i.e., definite integral is independent of the symbol of variable of integration.

3. $\int_a^b f(x)dx = \int_a^c f(x)dx = \int_c^b f(x)dx, a < c < b$

4. $\int_0^a f(x)dx = \int_0^a f(a-x)dx, a > 0$

5. $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & iff(-x)=f(x) \\ 0, & iff(-x)=-f(x) \end{cases}$

6. $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, & iff(2a-x)=f(x) \\ 0, & iff(2a-x)=-f(x) \end{cases}$

Example:- Evaluate $\int_0^1 x tan^{-1}x \, dx$

Solution:-We have, $\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{dx}{x^2+1}$

$=\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x$

$=\frac{(x^2+1)}{2}\tan^{-1}x - \frac{x}{2}$

$\therefore \int_0^1 x\tan^{-1}x\,dx = \left[\frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2}\right]_0^1$

$=\tan^{-1}1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$

Example:- Evaluate $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$

Solution:-Let I=$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx$

$=\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$=\int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx$

$\therefore$ 2I=I+I=$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}\,dx = \int_0^{\pi/2} \frac{(sinx+\cos x)}{(sinx+cosx)}\,dx$

$=\int_0^{\pi/2} dx = x]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore$ I=$\frac{\pi}{4}$

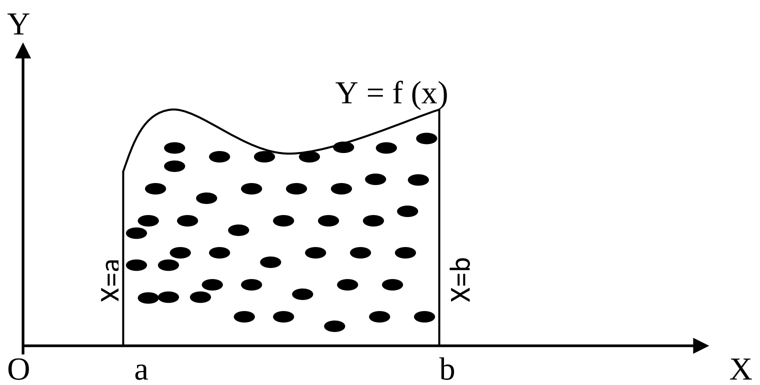
$\therefore \int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx = \frac{\pi}{4}$

AREA UNDER PLANE CURVES

DEFINITION-1:-

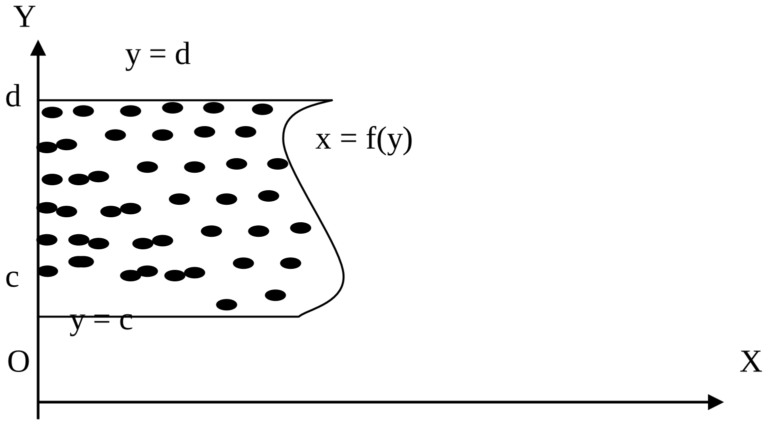
Area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a, x = b$ is defined by

Area=$\left|\int_a^b y dx\right| = \left|\int_a^b f(x)dx\right|$



DEFINITION-2:-Area of the region bounded by the curve $x = f(y)$, the Y-axis and the lines $y = c, y = d$ is defined by

Area=$\left|\int_c^d x dy\right| = \left|\int_c^d f(y) dy\right|$



Example:-Find the area of the region bounded by the curve $y = e^{3x}$, x-axis and the lines $x = 4, x = 2$.

Solution:-The required area is defined by

$$A=\int_2^4 e^{3x} dx = \frac{1}{3} e^{3x}\Big]_2^4 = \frac{1}{3}(e^{12x} - e^{6x})$$

Example:-Find the area of the region bounded by the curve $xy = a^2$, y-axis and the lines $y = 2, y = 3$

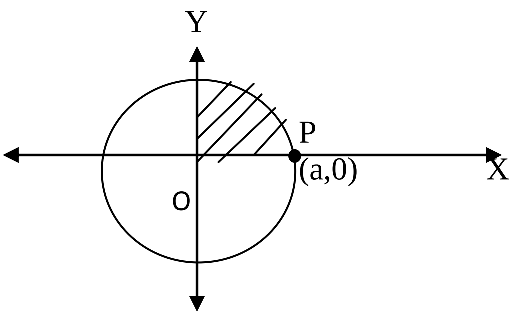
Solution:- $We\ have,\ xy = a^2 \Rightarrow x = \frac{a^2}{y}$

∴ The required area is defined by

$$A=\int_2^3 x dy = a^2 \int_2^3 \frac{dy}{y} = [a^2 \ln y]_2^3 = a^2(\ln 3 - \ln 2) = a^2 \ln\left(\frac{3}{2}\right)$$

Example:-Find the area of the circle $x^2 + y^2 = a^2$

Solution:-We observe that, $y = \sqrt{a^2 - x^2}$ in the first quadrant.



∴ The area of the circle in the first quadrant is defined by,

$A_1 = \int_0^a \sqrt{a^2 - x^2}\, dx$

As the circle is symmetrically situated about both X −axis and Y −axis, the area of the circle is defined by,

A=4 $\int_0^a \sqrt{a^2-x^2}\,dx$

=4 $\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right|_0^a$

= $4\frac{a^2}{2}sin^{-1}1 = 2a^2\frac{\pi}{2} = \pi a^2$.

DIFFERENTIAL EQUATIONS

DEFINITION:-An equation containing an independent variable (x), dependent variable (y) and differential co-efficients of dependent variable with respect to independent variable is called a differential equation.

For distance,

1. $\frac{dy}{dx} = \sin x + \cos x$
2. $\frac{dy}{dx} + 2xy = x^3$
3. $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION

The order of a differential equation is the order of the highest order derivative appearing in the equation.

Example:-In the equation, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$,

The order of highest order derivative is 2. So, it is a differential equation of order 2.

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the integral power of the highest order derivative occurring in the differential equation, after the equation has been expressed in a form free from radicals and fractions.

Example:-Consider the differential equation $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$

In this equation the power of highest order derivative is 1.So, it is a differential equation of degree 1.

Example:-Find the order and degree of the differential equation

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = K\frac{d^2y}{dx^2}$$

Solution:- By squaring both sides, the given differential equation can be written as $K^2\left(\frac{d^2y}{dx^2}\right)^2 - \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = 0$

The order of highest order derivative is 2.So, its order is 2.
Also, the power of the highest order derivative is 2.So, its degree is 2.

FORMATION OF A DIFFERENTIAL EQUATION

An ordinary differential equation is formed by eliminating certain arbitrary constants from a relation in the independent variable, dependent variable and constants.

Example:-Form the differential equation of the family of curves $y = a\sin(bx + c)$, a and c being parameters.

Solution:-We have $y = a\sin(bx + c)$ -------------(1)

Differentiating (1) w.r.t. x, we get

$\frac{dy}{dx} = ab\cos(bx + c)$ ----------------------(2)

Differentiating (2) w.r.t. x, we get

$\frac{d^2y}{dx^2} = -ab^2\sin(bx + c)$ -------------------(3)

Using (1) and (3), we get

$$\frac{d^2y}{dx^2} = -b^2y$$

$$\therefore \quad \frac{d^2y}{dx^2} + b^2y = 0$$

This is the required differential equation.

Example:-Form the differential equation by eliminating the arbitrary constant in $y = Atan^{-1}x$.

Solution:-We have, $y = Atan^{-1}x$ ------------------(1)

Differentiating (1) w.r.t. x, we get

$\frac{dy}{dx} = \frac{A}{1+x^2}$ -------------------(2)

Using (1) and (2), we get

$$\frac{dy}{dx} = \frac{y}{(1+x^2)tan^{-1}x}$$

$$\therefore \ (1 + x^2)tan^{-1}x\frac{dy}{dx} = y$$

This is the required differential equation.

SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is a relation $(like\ y = f(x)\ or\ f(x,y) = 0)$between the variables which satisfies the given differential equation.

GENERAL SOLUTION

The general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

PARTICULAR SOLUTION

A particular solution is that which can be obtained from the general solution by giving particular values to the arbitrary constants.

SOLUTION OF FIRST ORDDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

We shall discuss some special methods to obtain the general solution of a first order and first degree differential equation.

1. Separation of variables

2. Linear Differential Equations

3. Exact Differential Equations

SEPARATION OF VARIABLES

If in a first order and first degree differential equation, it is possible to separate all functions of x and dx on one side, and all functions of y and dy on the other side of the equation, then the variables are said to be separable.Thus the general form of such an equation is $f(y)dy = \mathrm{g}(x)dx$

Then, Integrating both sides, we get

$\int f(y)dy = \int \mathrm{g}(x)dx + C$ as its solution.

Example:-Obtain the general solution of the differential equation

$$9y\frac{dy}{dx} + 4x = 0$$

Solution:- We have, $9y\frac{dy}{dx} + 4x = 0$

$$\Rightarrow 9y\frac{dy}{dx} = -4x$$

$$\Rightarrow 9y\,dy = -4x\,dx$$

Integrating both sides, we get

$$9\int ydy = -4\int xdx$$

$$\Rightarrow \frac{9}{2}\cdot y^2 = \frac{-4}{2}x^2 + K$$

$\Rightarrow 9y^2 = -4x^2 + C$ (Where C=2K)

$$\Rightarrow 4x^2 + 9y^2 = C$$

This is the required solution

LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear, if the dependent variable and its differential co-efficients occurring in the equation are of first degree only and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q, \quad \text{-----------------(1)}$$

Where P and Q are functions of x.

To solve linear differential equation of the form (1),
at first find the Integrating factor = $e^{\int P\,dx}$ -------------------(2)

It is important to remember that

I.F = $e^{\int P \cdot dx}$

Then, the general solution of the differential equation (1) is

$$y.(I.F) = \int Q.(I.F)dx + C \quad \text{-------------------(3)}$$

Example:-Solve $\frac{dy}{dx} + y\sec x = \tan x$

Solution:-The given differential equation is

$$\frac{dy}{dx} + (\sec x)y = \tan x \quad \text{-----------------(1)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \text{ , where } P = \sec x \text{ and } Q = \tan x$$

∴ I.F= $e^{\int P \cdot dx} = e^{\int \sec x dx} = e^{\ln(\sec x + \tan x)}$

So, I.F=$\sec x + \tan x$

∴ The general solution of the equation (1) is

$$y.(I.F) = \int Q(I.F)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int \tan x (\sec x + \tan x)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \ \sec x + tan^2 x)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \ \sec x + sec^2 x - 1)dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \sec x + \tan x - x + C$$

This is the required solution.

Example:-Solve:$(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

Solution:-The given differential equation can be written as

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2$$

$$\Rightarrow \frac{dy}{dx}+\frac{2x}{1+x^2}\cdot y=\frac{4x^2}{1+x^2} \quad \text{----------------(1)}$$

This is a linear equation of the form $\frac{dy}{dx}+Py=Q$,

Where $P=\frac{2x}{1+x^2}$ and $Q=\frac{4x^2}{1+x^2}$

We have, I.F = $e^{\int P\cdot dx}$ = $e^{\int 2x/(1+x^2)dx}$ = $e^{\ln(1+x^2)}=1+x^2$ ---------------(2)

∴The general solution of the given differential equation (1) is

$y.(I.F)=\int Q.(I.F)dx+C$

$\Rightarrow \quad y(1+x^2)=\int\frac{4x^2}{1+x^2}.(1+x^2)dx+C$

$\Rightarrow \quad y(1+x^2)=4\int x^2dx+C$

$\Rightarrow \quad y(1+x^2)=\frac{4}{3}x^3+C$

This is the required solution

EXACT DIFFERENTIAL EQUATIONS

DEFINITION:- A differential equation of the form
$M(x,y)dx+N(x,y)dy=0$ is said to be exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

METHOD OF SOLUTION:-

The general solution of an exact differentia equation $Mdx+Ndy=0$ is

$$\int Mdx+\int(terms\ of\ N\ not\ containing\ x)dy=C,$$

(y=constant)

Provided $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Example:- Solve;$(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$.

Solution:-The given differential equation is of the form $Mdx+Ndy=0$.

Where, $M=x^2-4xy-2y^2$ and $N=y^2-4xy-2x^2$

We have $\frac{\partial M}{\partial y}=-4x-4y=\frac{\partial N}{\partial x}$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so the given differential equation is exact.

∴ The general solution of the given exact differential equation is

$$\int Mdx+\int(terms\ of\ N\ free\ from\ x)dy=C$$

(y=constant)

$\Rightarrow \quad \int(x^2-4xy-2y^2)dx+\int y^2dy=C$

(y=constant)

$\Rightarrow \quad \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = C$
$\Rightarrow \quad x^3 - 6x^2y - 6xy^2 + y^3 = C.$
This is the required solution.

Example:- Solve;$(x^2 - ay)dx = (ax - y^2)dy$

Solution:-The given differential equation can be written as

$(x^2 - ay)dx + (y^2 - ax)dy = 0$ ----------------(1)
Which is of the form $Mdx + Ndy = 0$,
Where,$M = x^2 - ay$ and $N = y^2 - ax$.
We have $\frac{\partial M}{\partial y} = -a \; and \; \frac{\partial N}{\partial x} = -a$
Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation (1) is exact.
∴ The solution of (1) is $\int (x^2 - ay)dx + \int y^2 dy = C$
(y=constant)
$\Rightarrow \frac{x^3}{3} - axy + \frac{y^3}{3} = C$
$\Rightarrow x^3 - 3axy + y^3 = C,$
Which is the required solution.

Example:- Solve; $ye^{xy}dx + (xe^{xy} + 2y)dy = 0.$

Solution:- The given differential equation is $ye^{xy}dx + (xe^{xy} + 2y)dy = 0,$

Which is of the form $Mdx + Ndy = 0.$
Where,$M = ye^{xy}$ and $N = xe^{xy} + 2y$
We have $\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} = \frac{\partial N}{\partial x}$
So the given equation is exact and its solution is
$\int ye^{xy}dx + \int 2ydy = C.$
(y=constant)
$\Rightarrow e^{xy} + y^2 = C$

Example:- Solve; $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$

Solution:- The given equation is of the form $Mdx + Ndy = 0,$
Where,$M = 3x^2 + 6xy^2$ and $N = 6x^2y + 4y^3$
We have $\frac{\partial M}{\partial y} = 12xy = \frac{\partial N}{\partial x}$.
So the given equation is exact and its solution is
$\int (3x^2 + 6xy^2)dx + \int (4y^3)dy = C$
(y=constant)
$\Rightarrow \frac{3x^3}{3} + \frac{6}{2}x^2y^2 + \frac{4}{4}y^4 = C$

$\Rightarrow x^3 + 3x^2y^2 + y^4 = C$
This is the required solution.